Bernoulli Equation

1 Sources of Aerodynamic Force

In previous section, we know there are four forces in equations of motion. The weight comes from gravity, the thrust comes from engine. How do the drag and lift come from? Based on the research, the lift comes from **pressure** and the drag comes from **shear stress**, which is caused by the viscosity of the fluid and the **no slip condition** at the surface.

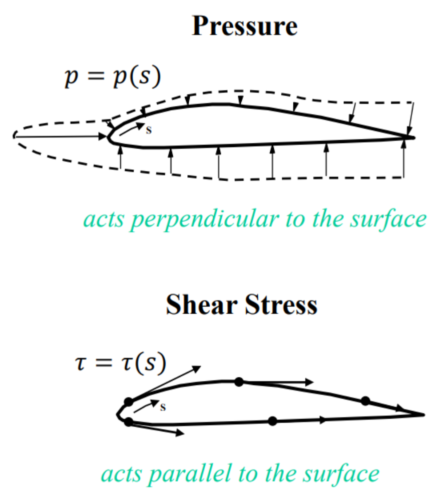

Figure 1: Source of Aerodynamic Forces

If we integrate around the surface of the body, we can get the total **resultant aerodynamic force (R)**:

$$\underline{\boldsymbol{R}} = \int\int_S p\underline{\boldsymbol{n}}dS + \int\int_S \tau\underline{\boldsymbol{k}}dS \tag{1}$$

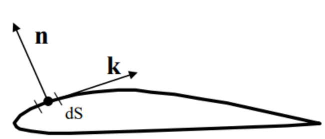

Figure 2: Resultant Aerodynamic Forces

2 Streamlines

2.1 Definition

A streamline is a line that is **tangential to the local velocity vector** of the flow field. It provides a means for flow pattern visualization based on **Eulerian** Approach. Assume the infinitesimal displacement along the streamline as $\underline{\boldsymbol{ds}}$, then we have:

$$\underline{\boldsymbol{ds}} \times \underline{\boldsymbol{U}} = \underline{\mathbf{0}} \tag{2}$$

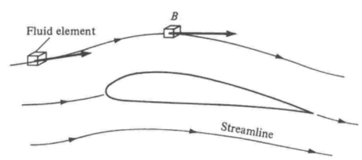

Figure 3: Streamlines

For steady flow, streamline pattern is invariant at different time, but in steady flow, it changes with time.

2.2 Characteristics

1. The **density** of streamlines in the vicinity (near region) of a point is proportional to the **velocity gradient** at that point.
2. The **mass** contained between any two streamlines is **conserved** throughout the flow field, so we do not have **normal velocity component (ρu)**
3. For a **rigid body** inside the flow field, a line on the **surface** of the body is a streamline.
4. The velocity at any point in the flow has **a single value and one direction** at the same instant, so two streamlines **can not intersect.**

2.3 Stream Function

Expand the definition expression of streamline:

$$(dx, dy, dz) \times (u, v, w) = \underline{\mathbf{0}} \tag{3}$$

$$(wdy - vdz, -(wdx - udz), vdx - udy) = \underline{\mathbf{0}} \tag{4}$$

Therefore, we have:

$$wdy - vdz = 0,\ \frac{w}{dz} = \frac{v}{dy} \tag{5}$$

$$wdx - udz = 0,\ \frac{w}{dz} = \frac{u}{dx} \tag{6}$$

$$vdx - udy = 0,\ \frac{v}{dy} = \frac{u}{dx} \tag{7}$$

Another way to do this: because the tangent of the streamline at any point in the direction of the velocity vector at that point, so we can directly write:

$$\frac{dx}{|\boldsymbol{ds}|} = \frac{u}{|\boldsymbol{U}|} \tag{8}$$

$$\frac{dy}{|\boldsymbol{ds}|} = \frac{v}{|\boldsymbol{U}|} \tag{9}$$

$$\frac{dz}{|\boldsymbol{ds}|} = \frac{w}{|\boldsymbol{U}|} \tag{10}$$

Therefore we get:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{11}$$

That's why normally for 2D flow, we define the stream function:

$$\psi = \psi(x, y) \tag{12}$$

With the velocity components:

$$u = \frac{\partial \psi}{\partial y} \tag{13}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{14}$$

This automatically satisfies the continuity equation. Recall the constant density continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{15}$$

Now we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = 0 \tag{16}$$

Along the streamline, if we take the derivative of stream function:

$$\partial \psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -vdx + udy = 0 \tag{17}$$

Therefore, **the stream function along a streamline has a constant value.**

3 Bernoulli Equation

In the previous section, we know the lift comes from pressure, but what is the mechanism for this process? Here we need to introduce **Bernoulli Equation**, which gives the relationship between pressure and velocity, to show how **integral of pressure** can generate lift.

3.1 Derivation

Recall the x component of NS Equations:

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x + (F_x)_{viscous} \tag{18}$$

Assume **inviscid flow, no body forces, steady flow:**

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{19}$$

If we multiply this equation by dx:

$$u\frac{\partial u}{\partial x}dx + v\frac{\partial u}{\partial y}dx + w\frac{\partial u}{\partial z}dx = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \tag{20}$$

Now use the streamline relations, we have:

$$u(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \tag{21}$$

$$udu = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \tag{22}$$

$$\frac{1}{2}d(u^2) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx \tag{23}$$

Similarly, we can get other components:

$$\frac{1}{2}d(v^2) = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \tag{24}$$

$$\frac{1}{2}d(w^2) = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz \tag{25}$$

Combine all of these equations, we get **Euler's Equation:**

$$\frac{1}{2}d(u^2 + v^2 + w^2) = -\frac{1}{\rho}(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz) \tag{26}$$

$$\frac{1}{2}d(\underline{\boldsymbol{U}}^2) = -\frac{dp}{\rho} \tag{27}$$

$$\boxed{dp = -\rho \underline{\boldsymbol{U}} d\underline{\boldsymbol{U}}} \tag{28}$$

Notice that Euler's Equation works for both **incompressible and compressible flow**. Now if we assume the flow is **incompressible, constant density**, and integrate between two points **along a streamline**, we can get **Bernoulli's Equation:**

$$\int_1^2 dp = -\rho \int_1^2 \underline{\boldsymbol{U}} d\underline{\boldsymbol{U}} \tag{29}$$

$$p_2 - p_1 = -\rho(\frac{U_2^2}{2} - \frac{U_1^2}{2}) \tag{30}$$

$$\boxed{p_1 + \frac{1}{2}\rho U_1^2 = p_2 + \frac{1}{2}\rho U_2^2} \tag{31}$$

3.2 Summary of Assumptions

1. Inviscid
2. Incompressible (density does not change with time or position)
3. No body forces
4. Neglect fluid friction
5. Idealized laminar flow
6. Barotropic ($\rho = f(p)$, but this function is a constant)
7. Continuous, steady flow along a streamline