Sizing Example 1

1 Given Information

The data is from an Air-to-Air Fighter (AAF).

Mission Phases & Segments		Performance Requirements
1-2	Takeoff Acceleration Rotation	2000 ft.. PA, 100°F, $S_{TO} = S_G + S_R \leq 1500$ ft.. $k_{TO} = 1.2$, μ_{TO}=0.05, max power V_{TO}, $t_R = 3$ s, max power
6-7	Supersonic Penetration And Escape Dash	1.5 M/ 30 k ft.., no afterburning (if possible)
7-8	Combat Turn 1 Turn 2 Acceleration	30,000 ft.. 1.6M, one 360 deg 5g sustained turn, with afterburning 0.9M, two 360 deg 5g sustained turns, with afterburning 0.8 → 1.6M, $t \leq 50$ s, max power
13-14	Landing Free roll Braking	2000 ft.. PA, 100˚F, $s_L = s_{FR} + s_B \leq 1500$ ft.. $k_{TD} = 1.15$, $t_{FR} = 3$ s, $\mu_B = 0.18$ Drag chute diameter 15.6 ft.., deployment ≤ 2.5 s
Max Mach Number		2.0M/40k, max power

Figure 1: AAF Specifications

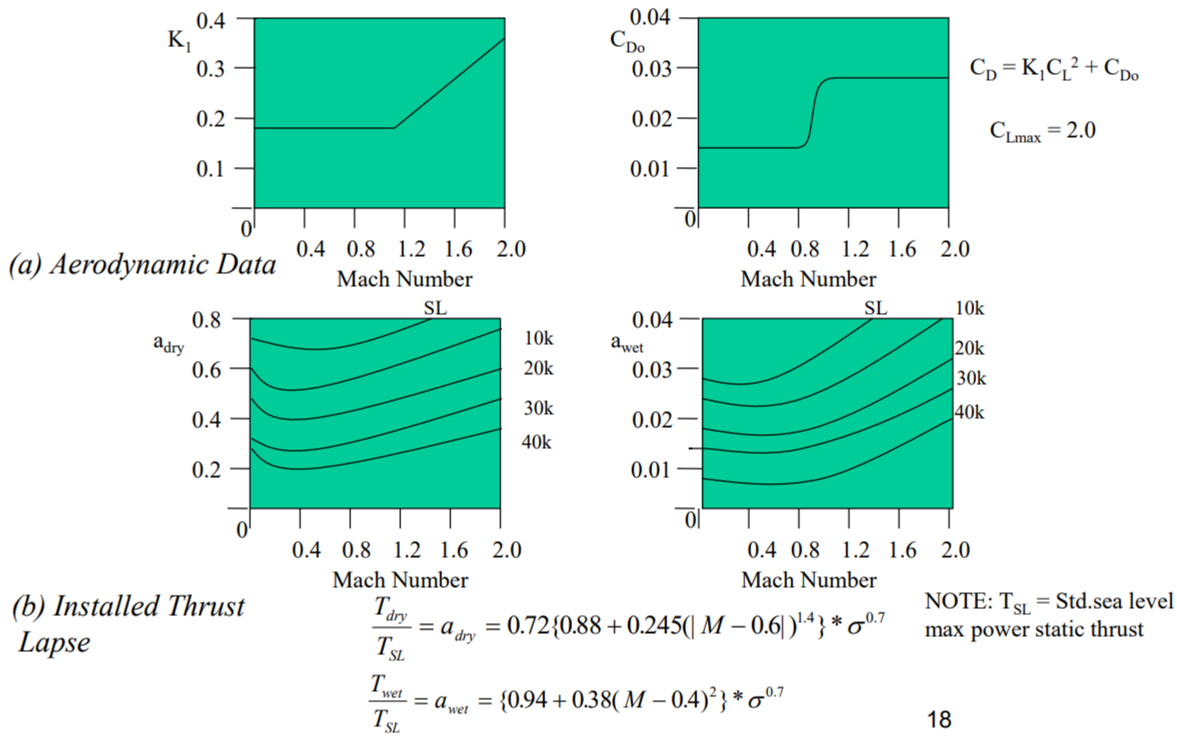

Figure 2: AAF Specifications

2 Constraint Analysis

2.1 Takeoff Constraint

2.1.1 Assumptions

1. Takeoff distance only includes ground roll distance and rotation distance
2. The airplane accelerated by thrust with **no resisting forces** in the ground roll
3. Thrust is balanced by drag forces during the constant velocity rotation

2.1.2 Constants

From the prompt, the altitude is 2000 ft and the temperature is 100 F, so from the atmosphere table:

1. $k_{TO} = 1.2$
2. $\beta = 1.0$
3. $\rho = 0.002047$ slugs/ft3
4. $g_0 = 32.17$ ft/s2
5. $C_{L,max} = 2.0$
6. $\alpha_{wet} = 0.8775$

7. $t_R = 3.0$ s

8. $s_{TO} = 1500$ ft

2.1.3 Calculation

Dry thrust refers to the normal operating thrust of a jet engine without the use of an afterburner. Wet thrust refers to the additional thrust produced when an afterburner is engaged. During the takeoff, max power is needed, so need to use wet thrust. Recall the energy equation section Case 5 and 6, when ignoring resisting forces (ξ_{TO} goes to zero), the ground roll distance is estimated as:

$$s_G = \frac{\beta^2}{\alpha_{wet}} \frac{k_{TO}^2}{\rho g_o C_{L_{max}} \frac{T_{SL}}{W_{TO}}} \left(\frac{W_{TO}}{S}\right) \tag{1}$$

The rotation distance could be expressed as:

$$s_R = t_R V_{TO} = t_R k_{TO} \sqrt{\left(\frac{2\beta}{\rho C_{L_{max}}}\right)\left(\frac{W_{TO}}{S}\right)} \tag{2}$$

Therefore, the takeoff distance in total is:

$$S_{TO} = \left(\frac{k_{TO}^2 \beta^2}{\rho^* g_0 * C_{L_{\max}} * \alpha_{\text{wet}}(\frac{T_{SL}}{W_{TO}})}\right)\left(\frac{W_{TO}}{S}\right) + t_R * k_{TO} \left(\frac{2\beta}{\rho C_{L_{\max}}}\right)^{\frac{1}{2}} \left(\frac{W_{TO}}{S}\right)^{\frac{1}{2}} \tag{3}$$

This could be rearranged as:

$$a\left(\frac{W_{TO}}{S}\right) + b\sqrt{\frac{W_{TO}}{S}} - c = 0 \tag{4}$$

After some magic algebra, the solution could be expressed as:

$$\left(\frac{W_{TO}}{S}\right) = \left\{\frac{-b + \sqrt{b^2 + 4ac}}{2a}\right\}^2 \tag{5}$$

Plug in the values:

$$a = \frac{12.47}{T_{SL}/W_{TO}} \tag{6}$$

$$b = 79.57 \tag{7}$$

$$c = 1500 \tag{8}$$

And the constraint curve data will be:

T_{SL}/W_{TO}	0.4	0.8	1.2	1.6	2.0	2.4
W_{TO}/S (lb./ft^2)	33.4	57.5	77.1	93.7	108	121

Figure 3: Takeoff Constraint Data

2.2 Maximum Mach Number Cruise Constraint

From the table, the max Mach number is 2, at altitude 40k with max power.

2.2.1 Assumptions

1. $\frac{dh}{dt} = 0$: constant altitude
2. $\frac{dV}{dt} = 0$: constant speed
3. $n = 1$: lift equals weight
4. $T = D$: thrust equals drag
5. $R = 0$: not on the ground
6. $h\&V$: values are given
7. K_2: small

2.2.2 Constants

1. $\beta = 0.78$
2. $(\alpha)_{M=2} = 0.7189$
3. $q = 1101$ lb/ft2
4. $K_1 = 0.36$
5. $C_{D_0} = 0.028$

Recall the energy equation section Case 1, the master equation will be:

$$\frac{T_{SL}}{W_{to}} = \frac{\beta}{\alpha}\left\{K_1\left(\frac{\beta}{q}\frac{W_{to}}{S}\right) + \frac{C_{D_0}}{\frac{\beta}{q}\frac{W_{to}}{S}}\right\} \tag{9}$$

Plug in the values:

$$\frac{T_{SL}}{W_{TO}} = 2.767 \times 10^{-4}\left(\frac{W_{TO}}{S}\right) + 42.88\left(\frac{W_{TO}}{S}\right) \tag{10}$$

2.3 Supersonic Cruise Constraint

Everything the same as max Mach number cruise, just the constants are different (not provided) and Mach number now is 1.5, altitude as 30K ft, no afterburner, so need to use α_{dry}.

2.4 Landing Constraint

2.4.1 Assumptions

1. The landing distance only include free roll distance and braking roll distance

2.4.2 Calculation

Recall the energy equation section Case 7, if the thrust reverser is large enough, we have the relations:

$$S_B = \frac{\beta}{\rho g \xi_L}\left(\frac{W_{TO}}{S}\right)\ln\left[1+\frac{\xi_L}{[\frac{(-\alpha)}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right)+\mu_B]\frac{C_{Lmax}}{k_{TD}^2}}\right] \tag{11}$$

If without the reverser ($\alpha = 0$), the equation will be:

$$S_B = \frac{\beta}{\rho g \xi_L}\left(\frac{W_{TO}}{S}\right)\ln\left[1+\frac{\xi_L}{\mu_B\frac{C_{Lmax}}{k_{TD}^2}}\right] \tag{12}$$

And the free roll distance is:

$$s_{FR} = t_{FR}k_{TD}\sqrt{\frac{2\beta}{\rho C_{L_{max}}}\left(\frac{W_{TO}}{S}\right)} \tag{13}$$

The constraint formula is:

$$s_L = s_{FR} + s_B \leq 1500 \tag{14}$$

Notice that these expressions do not have the term $\frac{T_{SL}}{W_{TO}}$, so the final value of $\frac{W_{TO}}{S}$ will be a constant, independent of thrust loading.

2.5 Design Point

Combine all the constraints, we have the design constraints graph as:

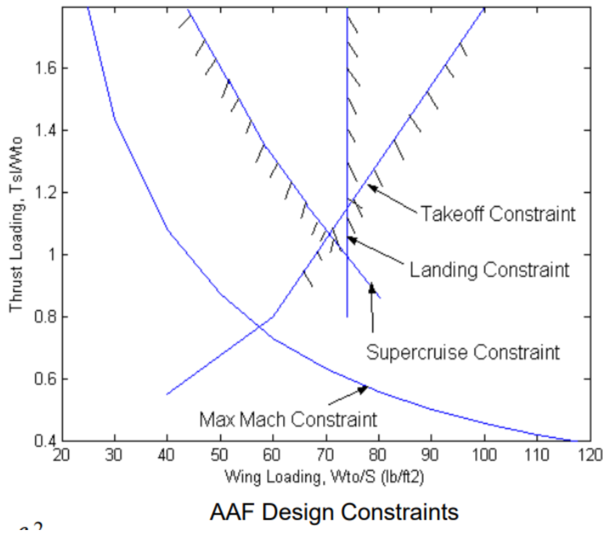

Figure 4: Design Constraints

Choose the first guess within the solution space:

$$\frac{T_{SL}}{W_{TO}} = 1.2 \tag{15}$$

$$\frac{W_{TO}}{S} = 64 \tag{16}$$

After selecting thrust loading and wing loading, take R = 0, the weight specific excess power could be expressed as:

$$P_s = V(\frac{\alpha}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right) - K_1\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) - K_2 - \left(\frac{C_{D_0}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)}\right)) \tag{17}$$

3 Detailed Calculation

Now, we need to calculate the relations between thrust loading and wing loading at each phase.

3.1 Phase 1-2: Takeoff

3.1.1 Assumptions and constants:

Variable	Value
β	1.0
ρ	0.002047 slug/ft^3
C_{Lmax}	2.0
k_{TO}	1.2
C_{D_0}	0.014
ξ_{TO}	0.36
K_1	0.18
σ	0.8613
α_{wet}	0.8775
μ_{TO}	0.05
t_R	3.0s
s_{TO}	1500 ft.

Table 1: Phase 1-2 Variables

3.1.2 Calculation

Without any approximation, the ground roll distance is expressed as:

$$s_G = -\frac{\beta\left(\frac{W_{TO}}{S}\right)}{\rho g_o \xi_{TO}} \ln\left[1 - \frac{\xi_{TO}}{\left[\frac{\alpha}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right) - \mu_{TO}\right]\frac{C_{Lmax}}{k_{TO}^2}}\right] \tag{18}$$

And the rotation distance is still:

$$s_R = t_R V_{TO} = t_R k_{TO}\sqrt{\left(\frac{2\beta}{\rho C_{L_{max}}}\right)\left(\frac{W_{TO}}{S}\right)} \tag{19}$$

The total distance is expressed as:

$$s_{TO} = s_G + s_R \tag{20}$$

Transfer the wing loading in the form:

$$\left(\frac{W_{TO}}{S}\right) = \left\{\frac{-b + \sqrt{b^2 + 4ac}}{2a}\right\}^2 \tag{21}$$

Then we get:

$$a = -\frac{\beta}{\rho g \xi_{TO}} \ln\left(1 - \frac{\xi_{TO}}{\left(\frac{\alpha}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right) - \mu_{TO}\right)\frac{C_{L_{MAX}}}{k_{TO}^2}}\right) \tag{22}$$

$$b = t_R k_{TO} \sqrt{\frac{2\beta}{\rho C_{L_{MAX}}}} \tag{23}$$

$$c = s_{TO} \tag{24}$$

Recall the expression of ξ_{TO}:

$$\xi_{TO} = (C_D + C_{DR} - \mu_{TO} C_L) \tag{25}$$

A conservative estimate for ξ_{TO} is by assuming $C_{DR} - \mu_{TO} C_L = 0$ and evaluating C_D at $C_L = C_{L,max}/k_{TO}^2$. After plugging in all the values, we have:

T_{SL}/W_{TO}	0.4	0.8	1.2	1.6	2.0
W_{TO}/S (lb./ft^2)	14.3	45.1	67.2	85.3	101

Table 2: Phase 1-2: Ratio Data

3.2 Phase 6-7 and 8-9: Supersonic Penetration and Escape Dash

3.2.1 Assumptions (most from Case 1):

1. $M = 1.5, 30000ft$
2. No afterburning
3. $\frac{dh}{dt} = 0$: constant altitude
4. $\frac{dV}{dt} = 0$: constant speed
5. $n = 1$: lift equals weight
6. $T = D$: thrust equals drag
7. $R = 0$: not on the ground
8. $h \& V$: values are given
9. K_2: small

3.2.2 Constants

Variable	Value
β	0.78
α_{dry}	0.3953
q	991.8 psf
σ	0.3747
K_1	0.28
C_{D0}	0.028

Table 3: Phase 6-7,8-9 Variables

3.2.3 Calculation

Same as Case 1:

$$\frac{T_{SL}}{W_{to}} = \frac{\beta}{\alpha}\left\{K_1\left(\frac{\beta}{q}\frac{W_{to}}{S}\right) + \frac{C_{D_0}}{\frac{\beta}{q}\frac{W_{to}}{S}}\right\} \tag{26}$$

And the final result curve:

T_{SL}/W_{TO}	2.35	1.77	1.2	0.913	0.746	0.638
W_{TO}/S (lb./ft^2)	20	40	60	80	100	120

Table 4: Phase 6-7, 8-9: Ratio Data

3.3 Phase 7-8: Combat Turn 1

3.3.1 Assumptions

1. $M = 1.6$
2. one 360 deg 5g sustained turns, with afterburning
3. $\frac{dh}{dt} = 0$: constant altitude
4. $\frac{dV}{dt} = 0$: constant speed
5. $R = 0$: not on the ground
6. $h\&V\&n$: values are given
7. $K_2 = 0$: pure parabolic drag polar

3.3.2 Constants

Variable	**Value**
β	0.78
α_{wet}	0.7481
q	1128 psf
σ	0.3747
n	5
K_1	0.3
C_{D0}	0.028

Table 5: Phase 7-8 Combat Turn 1 Variables

3.3.3 Calculation

Same as the Case 3, the expression is:

$$\frac{T_{SL}}{W_{to}} = \frac{\beta}{\alpha}\left\{K_1 n^2 \frac{\beta}{q}\left(\frac{W_{to}}{S}\right) + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{to}}{S}\right)}\right\} \tag{27}$$

Plug in all the values:

T_{SL}/W_{TO}	2.22	1.27	1.03	0.96	0.963	1
W_{TO}/S (lb./ft^2)	20	40	60	80	100	120

Table 6: Phase 7-8 Combat Turn 1: Ratio Data

3.4 Phase 7-8: Combat Turn 2

3.4.1 Assumptions

Everything is the same as combat turn 1, except:

1. $M = 1.6$
2. two 360 deg 5g sustained turns, with afterburning

3.4.2 Constants

Variable	Value
β	0.78
α_{wet}	0.5206
q	357 psf
σ	0.3747
n	5
K_1	0.18
C_{D0}	0.018

Table 7: Phase 7-8 Combat Turn 2 Variables

3.4.3 Calculation

The equation is the same as combat turn 1. The results are shown below:

T_{SL}/W_{TO}	0.91	0.9	1.09	1.33	1.6	1.87
W_{TO}/S (lb./ft^2)	20	40	60	80	100	120

Table 8: Phase 7-8 Combat Turn 2: Ratio Data

3.5 Phase 7-8: Horizontal Acceleration

3.5.1 Assumptions

1. $M = 0.8$ to $M = 1.6$
2. $t \leq 50$s
3. Max power

3.5.2 Constants

Notice all the values are gotten based on the mean Mach number 1.2.

Variable	**Value**
β	0.78
α_{wet}	0.5952
q	634.7 psf
σ	0.3747
n	5
K_1	0.23
C_{D0}	0.025
a	994.8 ft/s
ΔM	0.8
Δt	50 s

Table 9: Phase 7-8 Combat Turn 2 Variables

3.5.3 Calculation

Using the equation from Case 4, change into Mach number version:

$$\left(\frac{T_{SL}}{W_{TO}}\right) = \frac{\beta}{\alpha}\left[K_1\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + \frac{C_{D_0}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} + \frac{a\Delta M}{g_0\Delta t}\right] \tag{28}$$

Plug in all the values:

T_{SL}/W_{TO}	2.1	1.38	1.15	1.04	0.973	0.933
W_{TO}/S (lb./ft^2)	20	40	60	80	100	120

Table 10: Phase 7-8 Horizontal Acceleration: Ratio Data

3.6 Phase 13-14: Landing

3.6.1 Assumptions

1. 2000 ft, 100F
2. $s_L \leq 1500ft$
3. deployment less than 2.5 s

3.6.2 Constants

Variable	**Value**
β	0.56
ρ	0.002047 slug/ft^3
C_{Lmax}	2.0
k_{TD}	1.15
C_D	0.2775
C_{D_0}	0.014
C_{DR}	0.5348
ξ_L	0.8123
K_1	0.18
σ	0.8613
α	0
μ_B	0.18
t_{FR}	3.0s
s_L	1500 ft.

Table 11: Phase 13-14 Variables

3.6.3 Calculation

The equations are the same as the constraint section, now transfer to abc form:

$$a = -\frac{\beta}{\rho g_0 \xi_L} \ln \left(1 + \frac{\xi_L}{[\mu_B + \left(-\frac{a}{\beta}\right)\left(\frac{T_{ST}}{W_{TO}}\right)]\frac{C_{Lmax}}{k_{TD}^2}} \right) \tag{29}$$

$$b = t_{FR} k_{TD} \sqrt{\frac{2\beta}{\rho C_{L_{MAX}}}} \tag{30}$$

$$c = s_L \tag{31}$$

A conservative estimate of ξ_L from C_D at 0.8 of touchdown lift coefficient. Finally we get the constant of wing loading:

$$\frac{W_{TO}}{S} = 70.5 lb/ft2 \tag{32}$$