

# Compressibility

## 1 Overview

Compressible flow refers to the flow of a fluid (usually a gas) in which the fluid's density changes significantly as it moves. In a compressible flow, the fluid's speed, pressure, temperature, and density are all interrelated, and changes in one of these parameters can affect the others. In details:

1. **Pressure:** The pressure exerted by a gas is the result of molecules colliding with the walls of the container. When the pressure changes, it indicates that the space for molecules to move is changing. If pressure increases, the gas is compressed, it means the container's volume is reduced so the number of molecules per unit volume increases, so the gas becomes denser.
2. **Temperature:** As the temperature increases, faster-moving molecules want more space, so the volume a gas occupies tends to expand as the gas temperature increases. **Then the density will decrease with increasing temperature, assuming the number of molecules remains constant.**

## 2 Compressibility

Compressibility is a measure of how much a substance (such as a fluid or solid) changes its volume in response to a change in pressure or temperature. It is a property that describes the ability of a material to be compressed or compacted.

Recall the expression of volume using temperature and pressure  $V = V(T, p)$ , assume the composition ( $n_i$ ) constant, then we have:

$$dV = \left. \frac{\partial V}{\partial T} \right|_p dT + \left. \frac{\partial V}{\partial p} \right|_T dp \quad (1)$$

Therefore we define the **Isobaric Compressibility** (with the unit as 1/temp) as:

$$\alpha = \frac{1}{V} \left. \frac{\partial V}{\partial T} \right|_p \quad (2)$$

And the **Isothermal Compressibility** (with the unit as 1/pressure) as:

$$K = -\frac{1}{V} \left. \frac{\partial V}{\partial p} \right|_T \quad (3)$$

And the **Isentropic Compressibility** (with the unit also as 1/pressure) as:

$$\beta = -\frac{1}{V} \frac{\partial V}{\partial p} \Big|_s \quad (4)$$

Notice that all the compressibility coefficients are intensive variable.

### 3 Application

#### 3.1 Perfect Gas

Recall the ideal gas law, we have:

$$V = \frac{n\bar{R}T}{p} \quad (5)$$

Therefore we have:

$$\frac{\partial V}{\partial T} \Big|_p = \frac{V}{T} \quad (6)$$

$$\frac{\partial V}{\partial p} \Big|_T = -\frac{V}{p} \quad (7)$$

So we can rewrite the compressibility as:

$$\alpha = \frac{1}{V} \frac{\partial V}{\partial T} \Big|_p = \frac{1}{T} \quad (8)$$

$$K = -\frac{1}{V} \frac{\partial V}{\partial p} \Big|_T = \frac{1}{p} \quad (9)$$

Re-visit the volume expression:

$$dV = \alpha V dT - K V dp \quad (10)$$

Based on the **reciprocity law**:

$$\frac{\partial \alpha}{\partial p} \Big|_T = -\frac{\partial K}{\partial T} \Big|_p \quad (11)$$

After integration:

$$\int_{V_1}^{V_2} \frac{dV}{V} = \int_{T_1}^{T_2} \alpha dT - \int_{p_1}^{p_2} K dp \quad (12)$$

If we assume  $\alpha$ ,  $K$  constant, then:

$$\ln \frac{V_2}{V_1} = \alpha \Delta T_{12} - K \Delta p_{12} \quad (13)$$

From **cyclic rule**:

$$\left. \frac{\partial p}{\partial T} \right|_V \left. \frac{\partial T}{\partial V} \right|_p \left. \frac{\partial V}{\partial p} \right|_T = -1 \quad (14)$$

Finally we can get:

$$\left. \frac{\partial p}{\partial T} \right|_V = \frac{\alpha}{K} \quad (15)$$

### 3.2 Specific Heats

Recall the expression of entropy:

$$dS = \left. \frac{\partial S}{\partial V} \right|_T dV + \left. \frac{\partial S}{\partial T} \right|_V dT \quad (16)$$

From Maxwell relations:

$$\left. \frac{\partial S}{\partial V} \right|_T = \left. \frac{\partial p}{\partial T} \right|_V \quad (17)$$

$$\left. \frac{\partial S}{\partial T} \right|_V = \frac{1}{T} \left. \frac{\partial U}{\partial T} \right|_V \quad (18)$$

Plug in we can get:

$$dS = \left. \frac{\partial p}{\partial T} \right|_V dV + \frac{C_V}{T} dT \quad (19)$$

Similarly we have:

$$dS = - \left. \frac{\partial V}{\partial T} \right|_p dp + \frac{C_p}{T} dT \quad (20)$$

Equate these two equations:

$$dp = \frac{C_p - C_V}{T(\partial V / \partial T)_p} dT - \frac{(\partial p / \partial T)|_V}{(\partial V / \partial T)|_p} dV = \left. \frac{\partial p}{\partial T} \right|_V dT + \left. \frac{\partial p}{\partial V} \right|_T dV \quad (21)$$

Therefore:

$$\left. \frac{\partial p}{\partial T} \right|_V = \frac{C_p - C_V}{T(\partial V / \partial T)_p} \quad (22)$$

Recall from previous section:

$$\left. \frac{\partial p}{\partial T} \right|_V = \frac{\alpha}{K} \quad (23)$$

$$C_p - C_V = \frac{\alpha}{K} T \left. \frac{\partial V}{\partial T} \right|_p = \frac{\alpha}{K} T \alpha V = T \frac{\alpha^2 V}{K} \quad (24)$$

Also recall from previous section, for the perfect gas:

$$\alpha = \frac{1}{T}, \quad K = \frac{1}{p} \quad (25)$$

Therefore:

$$C_p - C_V = T \frac{(1/T)^2 V}{1/p} = \frac{pV}{T} = n\bar{R} \quad (26)$$

Some remarks:

1. For all stable substances, we have  $K > 0$ , so we can conclude that  $C_p \geq C_V$
2. If  $\alpha = 0$ , then we have  $C_p = C_V$ , therefore if  $T$  approaches 0, then we have  $C_p$  approaches  $C_V$