Isentropic Flow (Area Change)

1 Governing Equations

Recall the master equations, we have the mass conservation:

$$\frac{d\rho}{\rho} + \frac{1}{2}\frac{d(u^2)}{u^2} + \frac{dA}{A} = 0 \tag{1}$$

And the momentum conservation:

$$\frac{\rho}{p}\frac{u^2}{2}\frac{du^2}{u^2} + \frac{\tau_x}{p}\frac{L_p}{A}dx + \frac{dp}{p} = 0 \tag{2}$$

And now we assume **no viscous stress/friction (reversible), and no other forces or work**, then we have $\tau_x = 0$, so:

$$\frac{1}{2}\frac{du^2}{u^2} = -\frac{dp}{\rho u^2} = \frac{du}{u} \tag{3}$$

Combine these two equations, we get:

$$\frac{d\rho}{\rho} - \frac{dp}{\rho u^2} + \frac{dA}{A} = 0 \tag{4}$$

$$\frac{dA}{A} = \frac{dp}{\rho u^2}(1 - \frac{u^2}{dp/d\rho}) \tag{5}$$

If we further assume adiabatic ($\partial q = 0$), then we get isentropic flow. Based on the definition of speed of sound:

$$a^2 = \frac{dp}{d\rho} = \frac{\partial p}{\partial \rho}|_s \tag{6}$$

Therefore:

$$\frac{dA}{A} = \frac{dp}{\rho u^2}(1 - M^2) = -\frac{du}{u}(1 - M^2) \tag{7}$$

Because this is derived using only mass, momentum conservation and speed of sound definition, so it is valid for all simple compressible substances.

2 Mach Number Dependence

2.1 Nozzle and Diffuser

From the equation above, we can conclude that:

1. **For subsonic ($M < 1$):** dA, dp have **same** signs, dA, du have **opposite** signs.
2. **For supersonic ($M > 1$):** dA, dp have **opposite** signs, dA, du have **Same** signs.
3. dp, du always have **opposite** signs.

Because of this, the nozzles and diffusers for subsonic and supersonic flow are **different**:

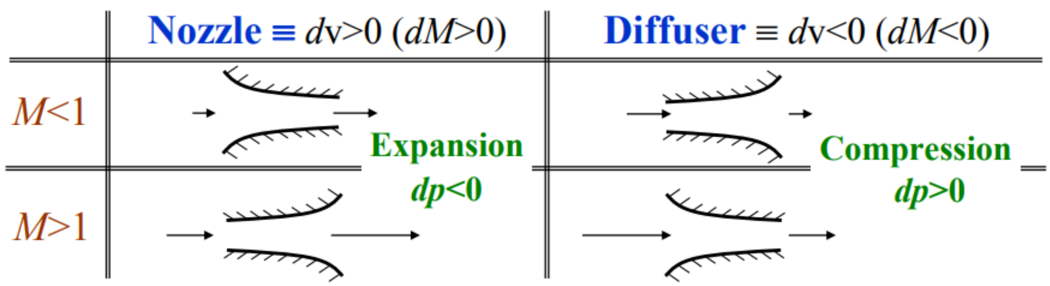

Figure 1: Subsonic and Supersonic Nozzles and Diffusers

2.2 Sonic Throat

If we want a **transition from subsonic to supersonic (or vice versa)**, we need to go through $M = 1$. The only two possible solutions are $dA = 0$ or $du = \infty$. However the later is not physical possible, so we must have a **maximum or minimum in area**.

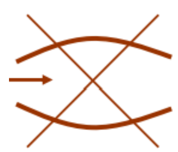

Figure 2: Maximum in Area

If we have maximum in area, when the subsonic flow goes in, it will decelerate and never reach unit Mach number. When the supersonic flow goes in, it will accelerate and never reach unit Mach number.

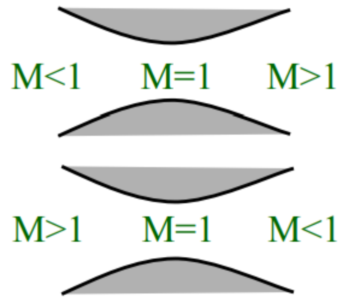

Figure 3: Throat

If we have minimum in area, **which is called throat**. When subsonic flow goes in it will **accelerate to unit Mach number**, then after passing the diverging section it will **keep accelerating to supersonic.** When supersonic flow goes in it will **decelerate to unit Mach number**, then after passing the diverging section it will **keep decelerating to subsonic.**

3 Area Ratio

For isentropic flow, we define the area at sonic point as A^*. Then based on mass conservation:

$$\rho u A = \rho^* u^* A^* \tag{8}$$

So we have:

$$\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{u^*}{a}\frac{a}{u} \tag{9}$$

Because A^* at sonic point, so $u^* = a^*$. Then recall the stagnation property:

$$\frac{\rho_o}{\rho} = (1 + \frac{\gamma - 1}{2}M^2)^{\frac{1}{\gamma-1}} \tag{10}$$

At sonic point, we have:

$$\frac{\rho_o}{\rho^*} = (1 + \frac{\gamma - 1}{2})^{\frac{1}{\gamma-1}} \tag{11}$$

So:

$$\frac{\rho^*}{\rho} = (\frac{1 + \frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}})^{\frac{1}{\gamma-1}} \tag{12}$$

Also we have:

$$\frac{a^*}{a} = \sqrt{\frac{\gamma R T^*}{\gamma R T}} = \sqrt{\frac{T^*}{T}} = \sqrt{\frac{T^*}{T_o}\frac{T_o}{T}} \tag{13}$$

Also recall the stagnation property:

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{14}$$

$$\frac{T_o}{T^*} = 1 + \frac{\gamma - 1}{2} \tag{15}$$

So we have:

$$\frac{a^*}{a} = \left(\frac{1 + \frac{\gamma-1}{2} M^2}{\frac{\gamma+1}{2}}\right)^{\frac{1}{2}} \tag{16}$$

Combine all the parts:

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1 + \frac{\gamma-1}{2} M^2}{\frac{\gamma+1}{2}}\right)^{\frac{1}{2}+\frac{1}{\gamma-1}} = \frac{1}{M}\left(\frac{1 + \frac{\gamma-1}{2} M^2}{\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{17}$$

Some numerical results are shown below:

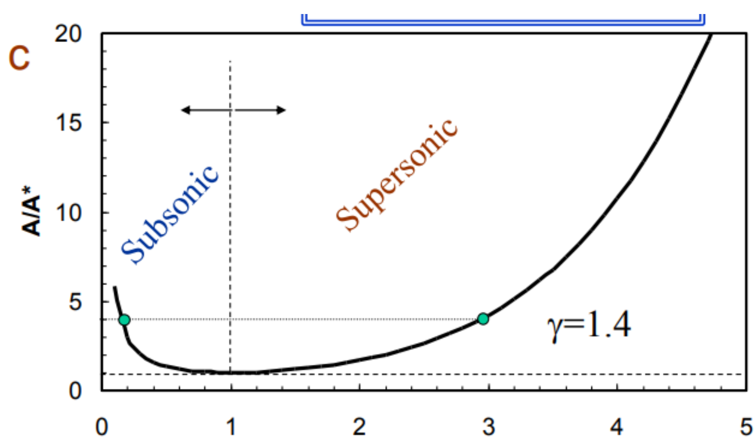

Figure 4: Area Ratio

Some remarks:

1. **There are always two isentropic solutions for given area ratio**, one subsonic and one supersonic.

2. A is always greater than A^*

3. The ratio of mass flux could be expressed as:

$$\frac{\dot{m}/A}{(\dot{m}/A)^*} = \frac{\rho u}{\rho^* u^*} = \frac{A^*}{A} \tag{18}$$

Because A is always greater, so **maximum mass flux at throat.**

4 Choked Flow

We can also express the mass flux using stagnation properties. Recall that:

$$\frac{\dot{m}}{A} = \rho u = \frac{p}{RT}Ma = \frac{p}{RT}M\sqrt{\gamma RT} = \frac{p}{\sqrt{RT}}\sqrt{\gamma}M \tag{19}$$

Using stagnation properties:

$$\frac{\dot{m}}{A} = \frac{p_o\frac{p}{p_o}}{\sqrt{RT_o\frac{T}{T_o}}}\sqrt{\gamma}M \tag{20}$$

Recall that:

$$\frac{T_o}{T} = 1 + \frac{\gamma-1}{2}M^2 \tag{21}$$

$$\frac{p_o}{p} = (1 + \frac{\gamma-1}{2}M^2)^{\frac{\gamma}{\gamma-1}} \tag{22}$$

Finally we have:

$$\frac{\dot{m}}{A} = \frac{p_o}{\sqrt{RT_o}}\frac{\sqrt{\gamma}M}{(1+\frac{\gamma-1}{2}M^2)^{\frac{\gamma+1}{2(\gamma-1)}}} = \frac{p_o}{\sqrt{RT_o}}f(\gamma, M) \tag{23}$$

For isentropic flow, all stagnation and sonic properties are constant, so is mass flow rate.

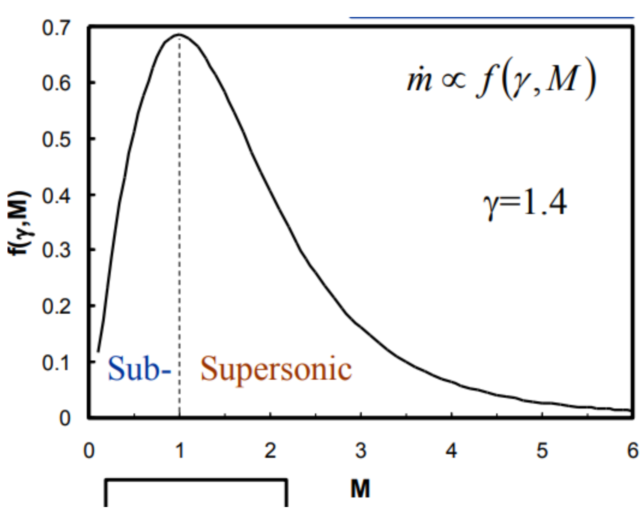

Figure 5: Mass Flow Rate

For **fixed stagnaton properties and flow area (but may not be isentropic)**, the maximum mass flow rate appears at $M = 1$. If the geometry has a sonic throat, then **we can not alter mass flow rate by changing downstream conditions (back pressure)**, which is called choked.

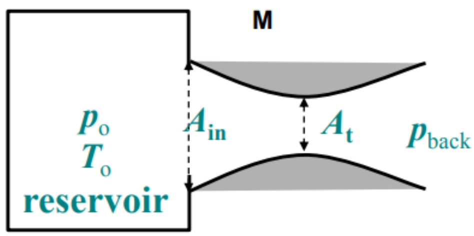

Figure 6: Choked Flow

Now we consider a situation. For nozzle with fixed stagnation properties and **initially sonic throat**:

1. **If we reduce throat Area:** A_{in}/A_t will increase, if the flow initially is subsonic, Mach number will increase. In this case **throat stays sonic**, and **mass flow rate will decrease** due to the decrease of throat area (RHS of mass flow rate equation is constant).

2. **If we increase throat area:** mass flow rate will increase due to same reason, eventually **throat can become not sonic**, so the flow will be unchoked.

Therefore we can conclude that the maximum mass flow rate will at sonic throat (M = 1):

$$\dot{m}_{max} = A^* \frac{p_o}{\sqrt{RT_o}} \frac{\sqrt{\gamma}}{(1 + \frac{\gamma-1}{2})^{\frac{\gamma+1}{2(\gamma-1)}}} = A^* \frac{p_o}{\sqrt{RT_o}} \sqrt{\gamma}(1 + \frac{\gamma - 1}{2})^{\frac{\gamma+1}{2(1-\gamma)}} \tag{24}$$

So if we want to **increase** mass flow rate, we can:

1. Increase throat area

2. Increase p_o or reduce T_o

The function $f(\gamma, 1)$ is near 0.7 for $\gamma = 1.4$, so we get the **rule of thumb for choked gas flow:**

$$\dot{m}_{max} \approx 0.7 \frac{p_o}{\sqrt{RT_o}} \tag{25}$$

5 Isentropic Nozzles

The function of nozzle is to **increase velocity of fluid and convert thermal energy to kinetic energy.** Normally we have two types of nozzles, including converging nozzle and converging-diverging (CD) nozzle.

5.1 Converging Nozzle

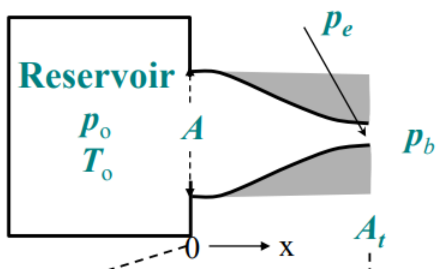

Figure 7: Converging Nozzle

If we assume choked and isentropic flow, there are large change in pressure and density as we approach throat, as shown below:

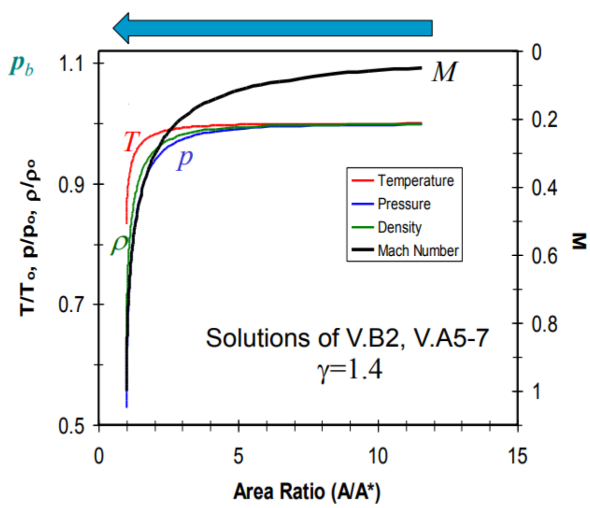

Figure 8: Converging Nozzle Profile

We use back pressure p_o/p_b to determine whether flow in nozzle gets choked (goes sonic).

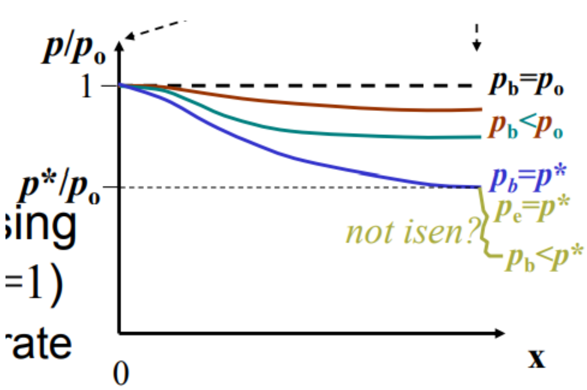

Figure 9: Converging Nozzle Back Pressure

Some remarks:

1. If $p_o = p_b$, then there will be no flow.
2. If we lower p_b from p_o, then Mach number at exit will keep **rising until flow is choked**
3. When $p_e = p^*$, mass flow rate reaches maximum.

So what is the **critical back pressure** required to go sonic in converging nozzle? We know at this situation:

$$\frac{p_b}{p_o} = \frac{p^*}{p_o} = \frac{1}{(1+\frac{\gamma-1}{2})^{\frac{\gamma}{\gamma-1}}} = (\frac{2}{\gamma+1})^{\frac{\gamma}{\gamma-1}} \tag{26}$$

When $\gamma = 1.4$, the approximate value is **0.528**.

5.2 Converging-Diverging Nozzle

The geometry and the profile are shown below:

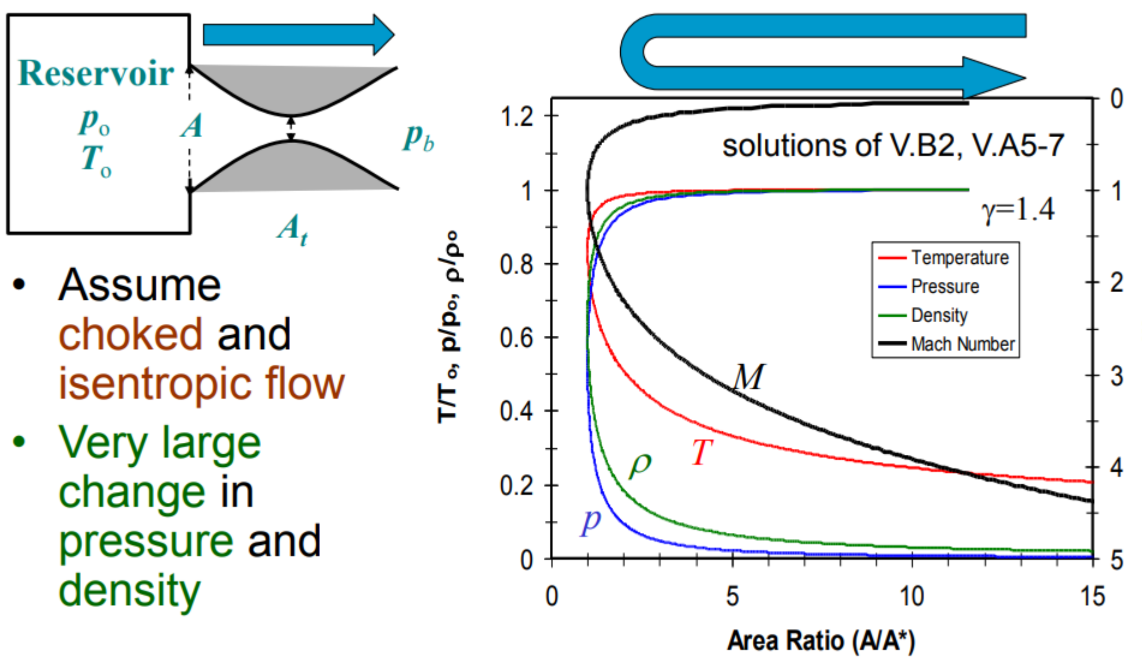

Figure 10: CD Nozzle

Similarly, we drop p_b from p_o:

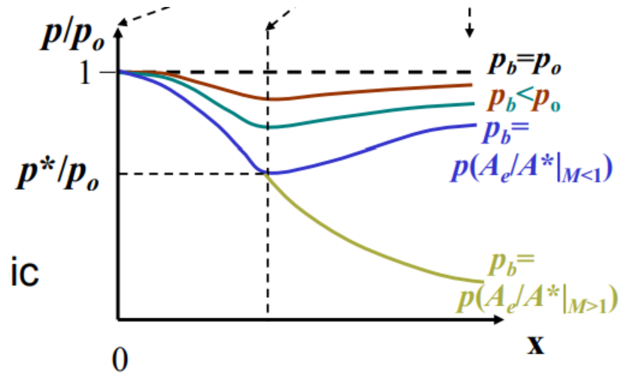

Figure 11: CD Nozzle Back Pressure

Then the exit Mach number M_e keeps rising and exit pressure p_e keeps dropping until flow is choked ($M_t = 1$). However, now **still subsonic at exit**.

After this point, if we lower p_b more, **nozzle will stay choked**. If lower enough, then we can get **supersonic exit Mach number**.