Master Equations

1 Problem Setup

Assume **Quasi-1D, steady, no body forces and negligible viscous work** flow, as shown below:

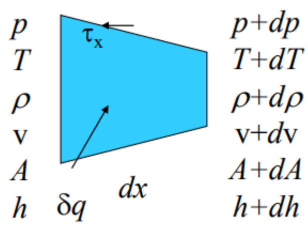

Figure 1: Problem Setup

2 Mass Conservation

Because we know the flow is steady:

$$\dot{m} = const = \rho u A \tag{1}$$

$$0 = d(\rho u A) \tag{2}$$

$$0 = uAd\rho + \rho A du + \rho u dA \tag{3}$$

Divide it by $\rho u A$:

$$0 = \frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = \frac{d\rho}{\rho} + \frac{1}{2}\frac{d(u^2)}{u^2} + \frac{dA}{A} \tag{4}$$

Recall the continuity equation:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \underline{\boldsymbol{u}}) = 0 \tag{5}$$

If steady flow:

$$\frac{D\rho}{Dt} = \frac{d\rho}{dt} + [\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}] \tag{6}$$

Because we assume 1D:

$$\frac{d(\rho u)}{dx} = 0 \tag{7}$$

Also due to constant mass flow rate, this in other words means:

$$\frac{dA}{dx} = 0 \tag{8}$$

Therefore we can conclude that quasi steady only valid if area changes slowly.

3 Momentum Conservation

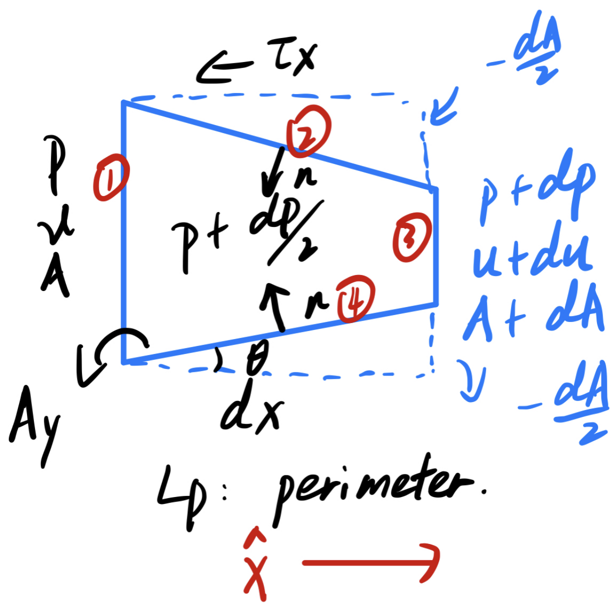

Figure 2: Momentum Conservation

Same conditions, recall the expression for momentum conservation:

$$\sum F = (\dot{m}u)_{net} \tag{9}$$

We define **right direction as positive direction**, then:

$$\sum F = -\tau_x L_p dx + F_p \tag{10}$$

Here τ_x is the shear stress, with the unit as F/A, and the unit of $L_p dx$ is A. For the force from pressure F_p, we divide it into 4 parts:

1. **Surface 1**:

$$(F_p)_{s1} = pA \tag{11}$$

2. **Surface 2,4**: This one is tricky. Because we only want the momentum in x direction, so we need to use the normal cross section, which is $-\frac{dA}{2}$, here dA is negative. Also, we assume the pressure acting at the middle point of the surface as $p+\frac{dp}{2}$. Therefore:

$$(F_p)_{s2+s4} = 2\times -(p+\frac{dp}{2})(-\frac{dA}{2}) = (p+\frac{dp}{2})dA \tag{12}$$

3. **Surface 3**:

$$(F_p)_{s3} = -(p+dp)(A+dA) \tag{13}$$

Now we add all the surfaces:

$$F_p = pA + (p+\frac{dp}{2})dA + -(p+dp)(A+dA) \tag{14}$$

Assume the second order term $dpdA$ as 0, then we have:

$$F_p = -Adp \tag{15}$$

Therefore, we have:

$$-\tau_x L_p dx - Adp = (\dot{m}u)_{net} = \dot{m}du = \rho uAdu \tag{16}$$

Rearrange and divide it by pA:

$$\frac{\rho}{p}udu + \frac{\tau_x}{p}\frac{L_p}{A}dx + \frac{dp}{p} = 0 \tag{17}$$

$$\frac{\rho}{p}\frac{u^2}{2}\frac{du^2}{u^2} + \frac{\tau_x}{p}\frac{L_p}{A}dx + \frac{dp}{p} = 0 \tag{18}$$

4 Energy Conservation

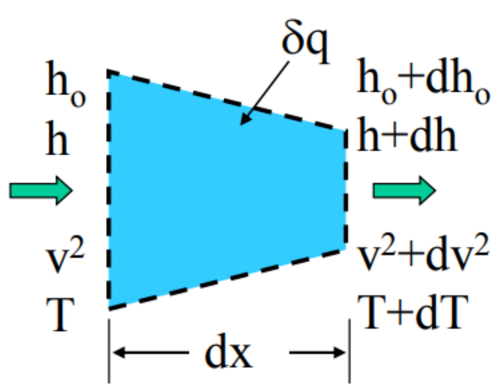

Figure 3: Energy Conservation

Recall the first law:

$$\partial q = dh_o = dh + d(\frac{u^2}{2}) = c_p dT + \frac{1}{2}du^2 \tag{19}$$

Therefore:

$$\frac{\partial q}{c_p T} = \frac{dT}{T} + \frac{1}{2c_p T}du^2 \tag{20}$$

Recall that:

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{21}$$

Finally we have:

$$\frac{dT}{T} = \frac{\partial q}{c_p T} - \frac{\gamma - 1}{2}\frac{u^2}{\gamma RT}\frac{du^2}{u^2} \tag{22}$$

5 Master Equations

Recall that:

$$M^2 = \frac{u^2}{\gamma RT},\ \frac{p}{\rho} = RT \tag{23}$$

We can rewrite the previous equations in terms of Mach number. **Mass:**

$$\frac{d\rho}{\rho} + \frac{1}{2}\frac{d(u^2)}{u^2} + \frac{dA}{A} = 0 \tag{24}$$

Momentum:

$$M^2\frac{\gamma}{2}\frac{du^2}{u^2} + \frac{\tau_x}{p}\frac{L_p}{A}dx + \frac{dp}{p} = 0 \tag{25}$$

Energy:

$$\frac{dT}{T} = \frac{\partial q}{c_p T} - \frac{\gamma - 1}{2}M^2\frac{du^2}{u^2} \tag{26}$$

Also from ideal gas law:

$$dp = d(\rho RT) = \rho R dT + RT d\rho \tag{27}$$

Therefore:

$$\frac{dp}{p} = \frac{\rho R}{p}dT + \frac{RT}{p}d\rho \tag{28}$$

$$\frac{dp}{p} = \frac{dT}{T} + \frac{d\rho}{\rho} \tag{29}$$

Similarly, we have:

$$d(u^2) = d(\gamma RTM^2) = \gamma M^2 RdT + M^2 RTd\gamma + \gamma RTdM^2 \tag{30}$$

Divide it by u^2, we have:

$$\frac{du^2}{u^2} = \frac{dT}{T} + \frac{d\gamma}{\gamma} + \frac{dM^2}{M^2} \tag{31}$$

Therefore, we have **5 equations and 5 unknowns (M, ρ, u, p, T)**, and 3 given inputs:

1. **Area change**: dA
2. **Shear stress/friction**: τ_x
3. **Heat transfer**: ∂q

We only get general analytic solutions if we **specify one input, and let the other two be 0**, so we have 5 equations 5 unknowns.