Normal Shock

1 Shock

1.1 Overview

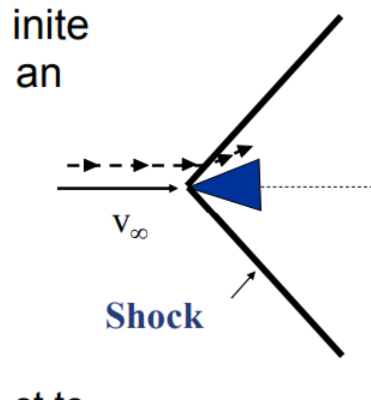

Figure 1: Shock

Recall the situation that **finite size body is moving faster than speed of sound**, then the fluid upstream **can not know body is coming before it gets there,** because the pressure information travels at speed of sound. So flow must suddenly adjust to presence of body, this adjustment occurs through the presence of a **shock**.

1.2 Sound Waves vs Shock Waves

Sound waves are weak, with minor density change ($d\rho/\rho \approx 0$). Also, they are **reversible and isentropic.**

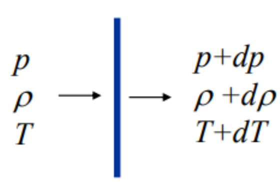

Figure 2: Sound Wave

Shock waves have strong compression ($\rho_2 > \rho_1$). The region for shock is very thin, so the changes in fluid properties are **nearly discontinuous.** The rapid change in properties is due to **internal viscous stresses**, so shock wave is **irreversible**. If we exclude the radiation, we can assume the shock wave is **adiabatic**. **Adiabatic + Irreversible = Nonisentropic.**

1.3 Formation

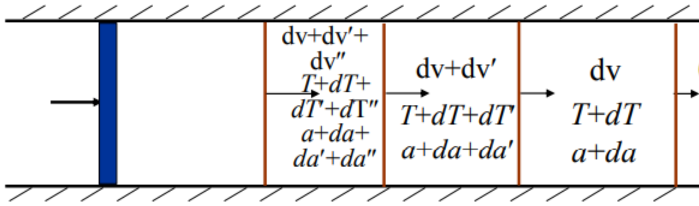

Figure 3: Shock Formation

Imagine we suddenly move the piston in tube in a very high speed, then it will produce **a series of discrete 1D compression waves**. Each pulse of piston produces **weak compression** wave travelling at **speed of sound** in moving gas in front of it. Each wave travels in **wake** of previous waves, even though they are all in speed of sound, **each wave travels slightly faster** because after the wave passing, the temperature will increase, so the velocity and speed of sound will increase too.

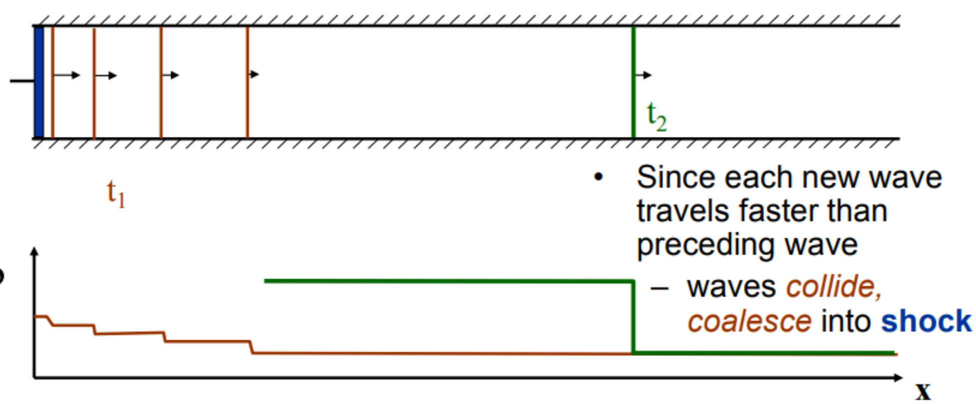

Figure 4: Coalescence of Compression Wave

For the **expansion waves**, for example successive waves see colder gas with lower a, **each new wave is slower than last.** This time, they **can not create shock** because they could not coalesce.

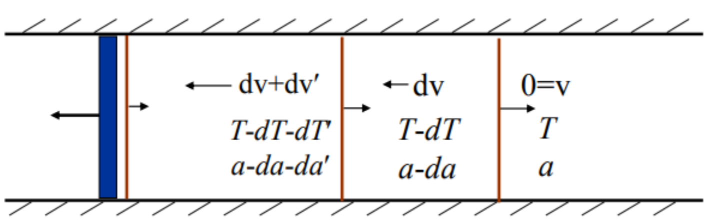

Figure 5: Expansion Wave

2 Normal Shock (Static Properties)

2.1 Definition

The shock with wave is **perpendicular to flow direction**.

2.2 Governing Equations

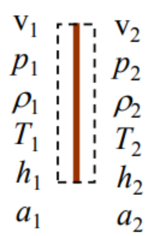

Figure 6: Problem Setup

Assumptions:

1. 1D
2. Stationary Shock (Steady)
3. Inviscid except inside shock
4. Adiabatic
5. Only flow work
6. Shock is non-equilibrium process internally, but we assume **flow before (1) and after (2) shock are in equilibrium.**

Then we have **mass conservation:**

$$\boxed{\frac{\dot{m}}{A} = \rho_1 u_1 = \rho_2 u_2} \tag{1}$$

Momentum Conservation:

$$p_1 A - p_2 A = \dot{m}_2 u_2 - \dot{m}_1 u_1 \tag{2}$$

So:

$$\boxed{p_1 - p_2 = \frac{\dot{m}}{A}(u_2 - u_1)} \tag{3}$$

Also:

$$p_1 A - p_2 A = \rho_2 u_2^2 A - \rho_1 u_1^2 A \tag{4}$$

So:

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2} \tag{5}$$

Energy Conservation:

$$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} = h_o} \tag{6}$$

Perfect Gas State Equations:

$$\boxed{p = \rho RT} \tag{7}$$

$$\boxed{dh = c_p dT} \tag{8}$$

$$\boxed{a^2 = \gamma RT} \tag{9}$$

So we have 6 equations, 6 unknowns (all the properties after the shock, properties before the shock are all known.)

2.3 Shock Hugoniot Equation

From energy equation:

$$h_2 - h_1 = \frac{1}{2}(u_1^2 - u_2^2) = \frac{1}{2}(u_1 - u_2)(u_1 + u_2) \tag{10}$$

Recall the momentum equation:

$$(u_1 - u_2) = \frac{p_2 - p_1}{\frac{\dot{m}}{A}} \tag{11}$$

And mass equation:

$$(u_1 + u_2) = (\frac{1}{\rho_1} + \frac{1}{\rho_2})\frac{\dot{m}}{A} \tag{12}$$

Therefore we can get the **Shock Hugoniot Equation:**

$$\boxed{h_2 - h_1 = \frac{1}{2}(p_2 - p_1)(\frac{1}{\rho_1} + \frac{1}{\rho_2})} \tag{13}$$

2.4 Entropy Change

For TPG/CPG, entropy state equation is:

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1} \tag{14}$$

Recall that:

$$c_v = \frac{R}{\gamma - 1} \tag{15}$$

Therefore:

$$\frac{s_2 - s_1}{c_v} = \ln [\frac{T_2}{T_1}(\frac{\rho_2}{\rho_1})^{-(\gamma-1)}] = \ln [\frac{p_2/\rho_2}{p_1/\rho_1}(\frac{\rho_2}{\rho_1})^{1-\gamma}] \tag{16}$$

Finally we have:

$$\boxed{\frac{s_2 - s_1}{c_v} = \ln [\frac{p_2}{p_1}(\frac{\rho_2}{\rho_1})^{-\gamma}]} \tag{17}$$

Entropy change as a function of pressure and density ratios across shock.

2.5 Velocity Ratio

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} \tag{18}$$

$$\boxed{\frac{u_2}{u_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}}} \tag{19}$$

2.6 Density Ratio

From mass conservation:

$$\rho_1 u_1 = \rho_2 u_2 \tag{20}$$

Therefore:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{M_1}{M_2}\sqrt{\frac{T_1}{T_2}}} \tag{21}$$

2.7 Temperature Ratio

$$h_{o1} = h_{o2} \rightarrow T_{o1} = T_{o2} \tag{22}$$

Therefore:

$$\frac{T_2}{T_1} = \frac{T_2/T_{o2}}{T_1/T_{o1}} \frac{T_{o2}}{T_{o1}} \tag{23}$$

$$\boxed{\frac{T_2}{T_1} = \frac{(1 + \frac{\gamma-1}{2} M_1^2)}{(1 + \frac{\gamma-1}{2} M_2^2)}} \tag{24}$$

2.8 Pressure Ratio

Recall the momentum equation:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{25}$$

Notice that:

$$\rho u^2 = \rho(M^2 a^2) = \rho(M^2 \gamma R T) = \rho(M^2 \gamma \frac{p}{\rho}) = p \gamma M^2 \tag{26}$$

Therefore:

$$p_1 + p_1 \gamma M_1^2 = p_2 + p_2 \gamma M_2^2 \tag{27}$$

$$\boxed{\frac{p_2}{p_1} = \frac{(1 + \gamma M_1^2)}{(1 + \gamma M_2^2)}} \tag{28}$$

2.9 Mach Number

Now the most tricky but most important relation appears. From mass conservation:

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} \tag{29}$$

For the LHS:

$$\frac{u_1}{u_2} = \frac{M_1 \sqrt{\gamma R T_1}}{M_2 \sqrt{\gamma R T_2}} \tag{30}$$

$$\frac{T_2}{T_1} = \frac{(1 + \frac{\gamma-1}{2} M_1^2)}{(1 + \frac{\gamma-1}{2} M_2^2)} \tag{31}$$

For the RHS:

$$\frac{\rho_2}{\rho_1} = \frac{p_2/RT_2}{p_1/RT_1} \tag{32}$$

$$\frac{p_2}{p_1} = \frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)} \tag{33}$$

Combining all these equation, after some magics, we get:

$$\boxed{\frac{M_2}{1+\gamma M_2^2}\sqrt{1+\frac{\gamma-1}{2}M_2^2} = \frac{M_1}{1+\gamma M_1^2}\sqrt{1+\frac{\gamma-1}{2}M_1^2}} \tag{34}$$

Solving this equation, we get:

$$\boxed{M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}} \tag{35}$$

In shock's reference frame, flow before normal shock (M_1) is always supersonic, the flow after normal shock (M_2) is always subsonic.

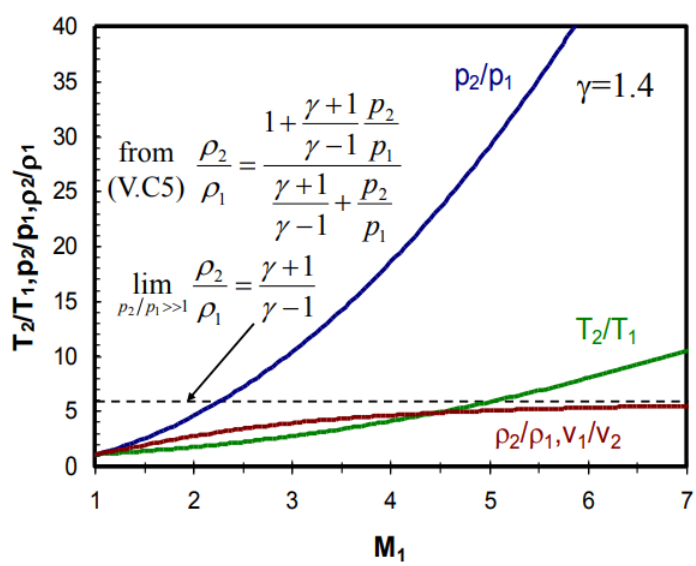

Figure 7: Property Ratios

Property ratios summary:

1. T, p, ρ **increase** across shock and ratios increase with M_1
2. u **decreases**
3. p ratio increase across normal shock is greatest static property change

3 Normal Shock (Stagnation Properties)

Now we want to know the stagnation properties relations.

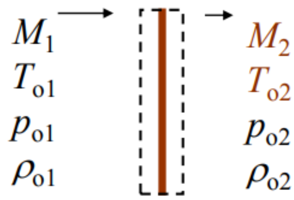

Figure 8: Stagnation Properties

3.1 Stagnation Temperature

Due to energy conservation, we know:

$$\boxed{T_{o2} = T_{o1}} \tag{36}$$

3.2 Stagnation Pressure

$$\frac{p_{o2}}{p_{o1}} = \frac{p_{o2}/p_2}{p_{o1}/p_1}\frac{p_2}{p_1} = \left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}}\frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)} \tag{37}$$

Recall that:

$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1} \tag{38}$$

After some magic math, we get:

$$\boxed{\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}} \tag{39}$$

$$\boxed{\frac{p_{o2}}{p_{o1}} = \left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{\frac{1}{1-\gamma}}} \tag{40}$$

There are also some other useful expressions. Recall that:

$$\frac{T_2}{T_1} = \frac{(1+\frac{\gamma-1}{2}M_1^2)}{(1+\frac{\gamma-1}{2}M_2^2)} \tag{41}$$

Therefore we have:

$$\boxed{\frac{p_{o2}}{p_{o1}} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}}\frac{p_2}{p_1}} \tag{42}$$

Also we know:

$$\frac{p_{o2}}{p1} = \frac{p_{o2}}{p2}\frac{p_2}{p_1} \tag{43}$$

So we have:

$$\boxed{\frac{p_{o2}}{p_1} = (1 + \frac{\gamma - 1}{2}M_2^2)^{\frac{\gamma}{\gamma-1}}\frac{p_2}{p_1}} \tag{44}$$

3.3 Stagnation Density

$$\frac{\rho_{o2}}{\rho_{o1}} = \frac{p_{o2}/RT_{o2}}{p_{o1}/RT_{o1}} \tag{45}$$

We know the stagnation temperature is constant, so:

$$\boxed{\frac{\rho_{o2}}{\rho_{o1}} = \frac{p_{o2}}{p_{o1}}} \tag{46}$$

3.4 Sonic Area Ratio

Recall the mass flow rate expression:

$$\cancelto{1}{\frac{\dot{m}_2}{\dot{m}_1}} = \frac{A_2^*}{A_1^*}\frac{p_{o2}/\sqrt{RT_{o2}}}{p_{o1}/\sqrt{RT_{o1}}}\cancelto{1}{\frac{f(\gamma)}{f(\gamma)}} \tag{47}$$

Therefore:

$$\boxed{\frac{A_2^*}{A_1^*} = \frac{p_{o1}}{p_{o2}}} \tag{48}$$

3.5 Summary

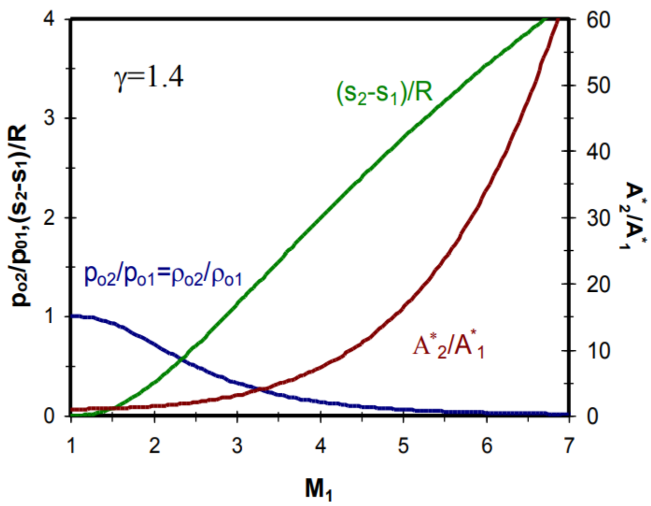

Figure 9: Stagnation Properties Ratio

Some remarks:

1. Stagnation temperature constant
2. Stagnation pressure and stagnation density **drop**
3. Entropy **increases**
4. **Sonic area increases**, so larger throat required after shock to reach sonic flow. (They have same mass flow rate, but after the shock the stagnation pressure drops, $\dot{m} \propto A^* p_o$)

4 Moving Normal Shock

So far, we focus on the changes across shock wave for the case of the shock **not moving**, which means observer 'sitting' on the shock, moving with shock:

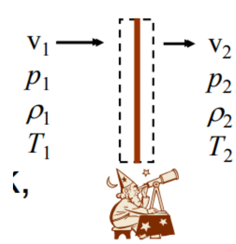

Figure 10: Stationary Shock

Now we want to consider the shock to be moving, so the observer not moving at same speed as shock:

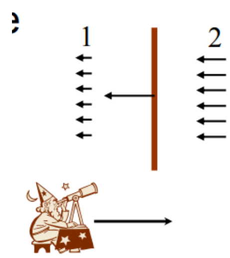

Figure 11: Moving Shock

To achieve this, we need to perform **Galilean Transform** to convert moving shock to stationary shock:

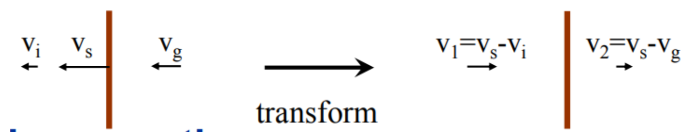

Figure 12: Galilean Transform

After the transformation, **static properties** are not affected, we can still use all the equations, but with:

$$M_1 = \frac{v_s - v_i}{a_i} \tag{49}$$

$$M_2 = \frac{v_s - v_g}{a_g} \tag{50}$$

However, the **stagnation properties** depend on velocity, not the same after transform.

5 Reflected Normal Shocks

When moving shock runs into a boundary, **change in boundary conditions must be transmitted back to flow**, which is called **reflected waves**.

5.1 Closed Tube

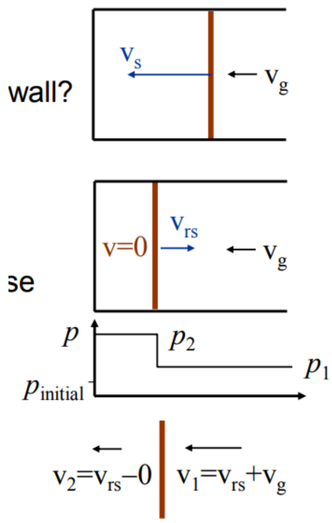

Figure 13: Closed Tube

While shock is heading towards wall, there is flow behind it. Because of the closed boundary, so no flow can go through wall. Therefore, **it must generate reflected wave that stops oncoming flow.** Then, flow is slowed down, p, ρ **increase, compression appears, it becomes reflected shock.**

In lab reference frame, $u = 0$ behind reflected wave, but in shock reference frame, we have $u_{rs} = u_2$, so using this relationship with known u_g, we can find u_{rs}.

5.2 Open Tube

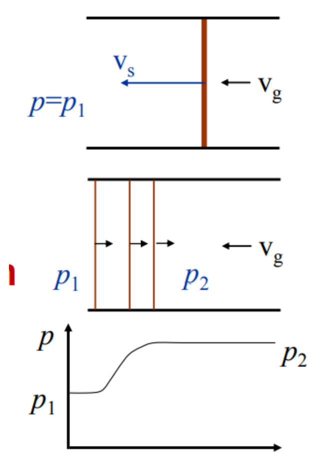

Figure 14: Open Tube

Now boundary condition is constant pressure at open end. Assume the outside pressure same as **pre-shock pressure** p_1, based on the shock relations we know $p_2 > p_1$, so the **reflection is expansion**. We already know there is no expansion shock, so reflection is smooth set of expansion waves.

6 Normal Shocks in CD Nozzles

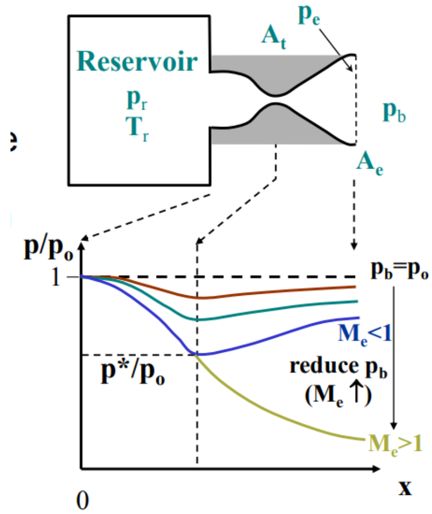

Figure 15: Shock in CD Nozzle

As the back pressure reduces from p_o, we will get **two isentropic solutions to get sonic flow at throat**: higher back pressure for subsonic, lower back pressure for supersonic.

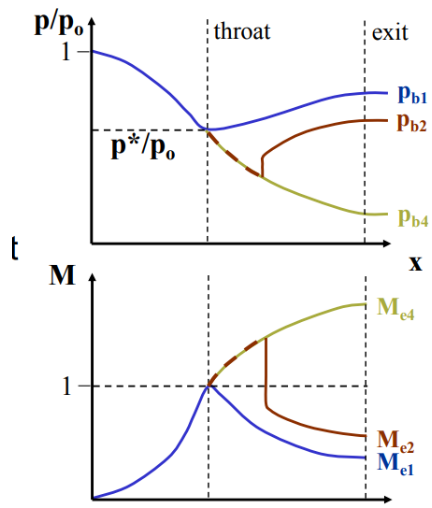

Figure 16: Nonisentropic Solution

Between these two isentropic solutions, we have **nonisentropic solutions.** When back pressure decreases until $p_b < p_{b1}$, flow starts to go supersonic after throat, before it reaches p_{b4}, **p must increase above supersonic isentropic case to math** p_b. Which means there will be **shock in diverging section.**

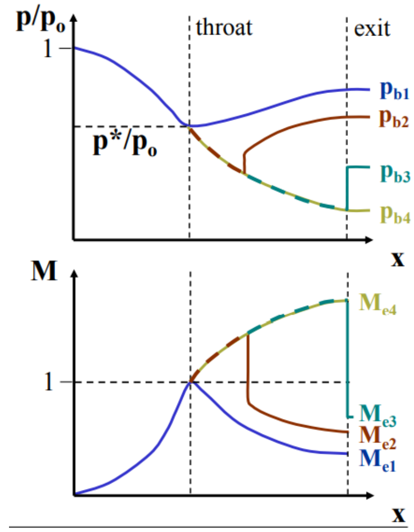

Figure 17: Shock Inside Nozzle

Now we want to know the range of back pressure will there be shock in nozzle, so we need to find the value **until shock occurs at exit plane.**

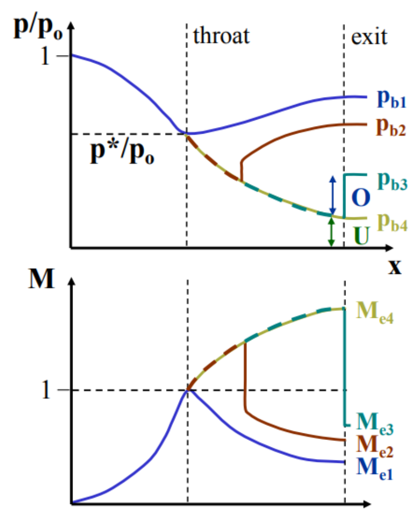

Figure 18: Over/Under Expanded Nozzles

If we keep dropping the back pressure to $p_b < p_{b3}$, then we have **isentropic flow up to exit and supersonic exhaust**, with shocks (but nor normal shock) outside nozzle. We define:

1. $p_{b4} < p_b < p_{b3}$: Overexpanded exhaust

1. $p_b < p_{b4}$: Underexpanded exhaust

7 Supersonic Windtunnels

7.1 Setup

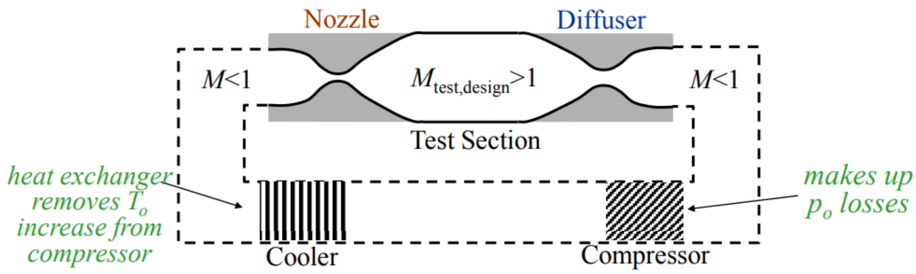

Figure 19: Supersonic Windtunnels

Normally, we use **closed circuit tunnel**, which requires less power to operate and does not have to accelerate flow as much. Because we need **subsonic** flow for compressor, diffuser is designed at the end (less pressure loss comparing with using shock).

7.2 Starting Problem

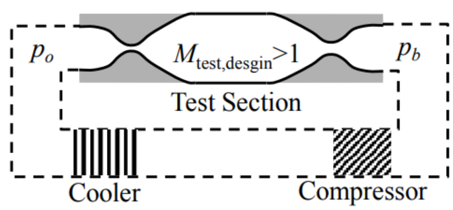

Figure 20: Starting Problem

So how do we actually start the wind tunnel? There is no initial velocity and uniform pressure throughout tunnel. We need to start tunnel by **changing** p_o/p_b **using compressor.** Recall the pressure profile:

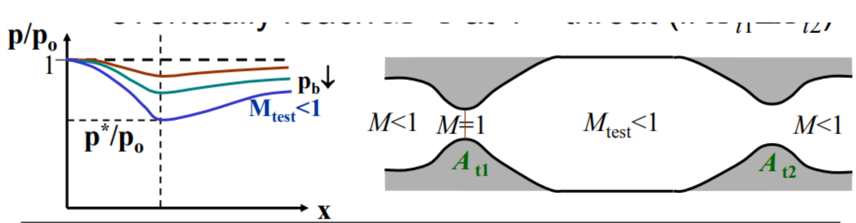

Figure 21: Pressure Profile

As we decrease p_b and increase p_o/p_b, we start with subsonic flow everywhere, eventually reach sonic at the first throat.

7.3 Starting Shock

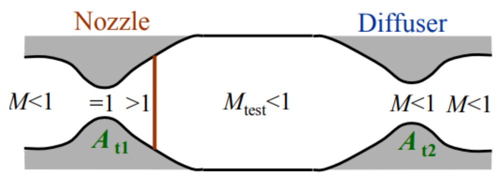

Figure 22: Starting Shock

If we decrease the back pressure more, then **normal shock moves into diverging section.** Recall the shock relations, A^* will increase across a shock. So if we want to get same mass flowrate, we need $A_{t2} > A_{t1}$. A_{t2} needs to satisfy the biggest p_o loss for strongest shock in test section.

7.4 Swallowing Shock

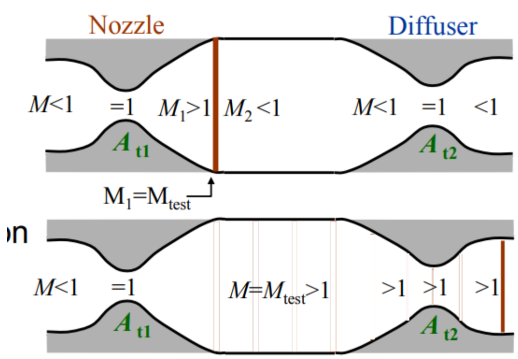

Figure 23: Swallowing Shock

If we decrease back pressure again, then shock will enter the test section. At this time, it will be sonic at the second throat. If we further decrease back pressure, we can get **shock to leave test section** and pass through second throat. This is also called **shock swallowed**.

If we want to run tunnel with lowest power requirements, then we require lowest p_o loss. Therefore, **the weakest shock is when it is at diffuser throat.** However, for stability, we want to operate the tunnel to let shock just downstream of throat.