Prandtl Meyer Wave

1 Overview

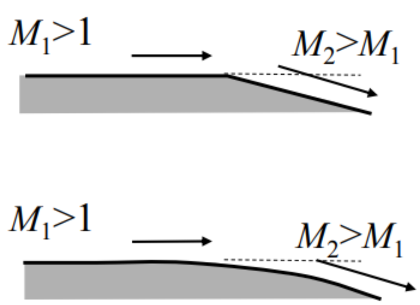

Figure 1: Expansion Case

In previous chapter, we examined supersonic over **sharp, concave corners/turns**, and **oblique shock** allows flow to make that turn. Now what if the turn is **convex or gradual?** Now the shock is impossible due to expansion.

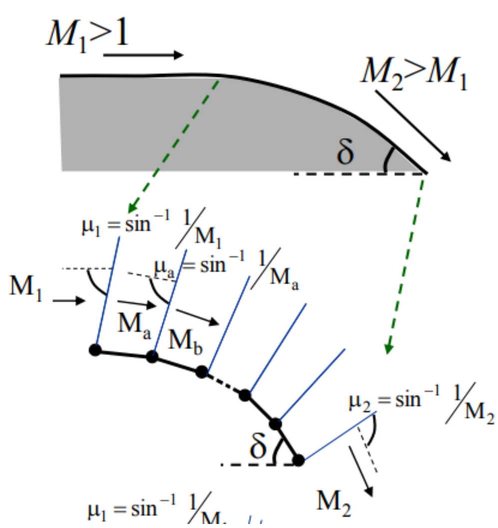

Figure 2: Gradual Turn

Actually, if we take a close look, the gradual turn **is made up of large umber of infinitesimal turns,** and each turn has infinitesimal flow change. Each turn is produced by **infinitesimal wave, which is called Mach wave.**

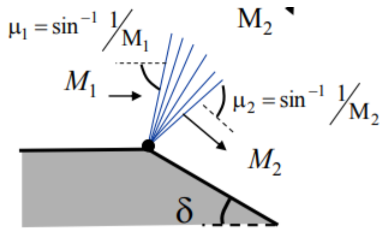

Figure 3: Wave Collapse

We assume the flow is **uniform and isentropic** between each turn, and the length between each is arbitrary. Therefore, it could be **zero length (sharp turn)** and waves collapse to one point. This is called **Prandtl Meyer Fan**. Notice that this is an expansion, so streamlines get farther apart.

2 Problem Setup

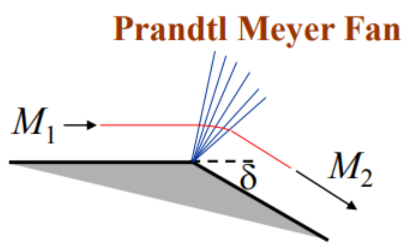

Figure 4: Problem Setup

2.1 Problem

Given **upstream conditions** (M_1) and **turning angle** (δ) to find **downstream conditions** M_2 and Mach number relations.

2.2 Equations

Mass, momentum, energy conservation, Mach number definition and state equations.

2.3 Assumptions

Steady flow, quasi-1D, **reversible and adiabatic (isentropic).**

3 Mach Relations

3.1 Relation Between Velocity and Angles

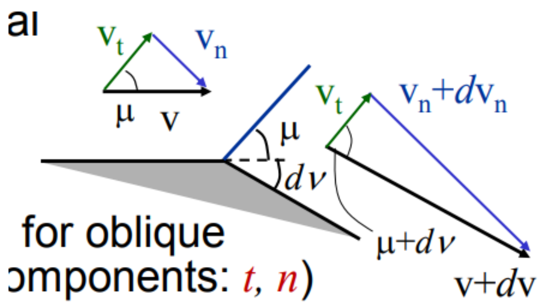

Figure 5: Infinitesimal Change

We can start with single Mach wave that expands supersonic flow through an **infinitesimal (differential) angle of magnitude** $d\nu$. Here we define ν as **Prandtl Meyer Angle.** Similar with oblique shocks approach, we divide the velocity into two components (t, n). Also due to **lack of pressure gradient tangent to wave** gives u_t is constant across wave.

Therefore, using this relation:

$$u_{t,upstream} = u_{t,downstream} \tag{1}$$

$$u \cos \mu = (u + du) \cos(\mu + d\nu) \tag{2}$$

$$u \cos \mu = (u + du)(\cos \mu \cos d\nu - \sin \mu \sin d\nu) \tag{3}$$

Based on the assumption that:

$$d\nu \to 0, \;\; dud\nu \to 0 \tag{4}$$

So we have:

$$\cos d\nu \to 1, \;\; \sin d\nu \to d\nu \tag{5}$$

$$u \cos \mu = u \cos \mu - ud\nu \sin \mu + du \cos \mu - dud\nu \sin \mu \tag{6}$$

Rearrange and simplify:

$$\frac{du}{u} = \frac{\sin \mu}{\cos \mu} d\nu \tag{7}$$

Recall the definition of Mach angle:

$$\sin \mu = \frac{1}{M} \tag{8}$$

And the trigonometric relationship:

$$\cos^2\mu = 1 - \sin^2\mu \tag{9}$$

Therefore we have:

$$\frac{du}{u} = \sqrt{\frac{1/M^2}{1-1/M^2}}d\nu \tag{10}$$

$$\boxed{\frac{du}{u} = \frac{1}{\sqrt{M^2-1}}d\nu} \tag{11}$$

3.2 Relation Between M and $d\nu$

Recall the definition of Mach number:

$$u = Ma \tag{12}$$

Therefore we take the derivative:

$$du = d(Ma) = Mda + adM \tag{13}$$

$$\frac{du}{u} = \frac{dM}{M} + \frac{da}{a} \tag{14}$$

Recall the speed of sound definition under TPG and CPG assumption:

$$a = \sqrt{\gamma RT} \tag{15}$$

Also take the derivative:

$$da = \sqrt{\gamma R}d\sqrt{T} \tag{16}$$

Plug back in we have:

$$\frac{du}{u} = \frac{dM}{M} + \frac{d\sqrt{T}}{\sqrt{T}} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T} \tag{17}$$

Recall the energy conservation:

$$T_o = T(1 + \frac{\gamma-1}{2}M^2) = const \tag{18}$$

Same approach for the derivative:

$$\frac{dT_o}{T_o} = \frac{dT}{T} + \frac{d(1+\frac{\gamma-1}{2}M^2)}{(1+\frac{\gamma-1}{2}M^2)} = 0 \tag{19}$$

Rearrange we can get:

$$\frac{dT}{T} = -\frac{d(1+\frac{\gamma-1}{2}M^2)}{(1+\frac{\gamma-1}{2}M^2)} = -\frac{M(\gamma-1)dM}{(1+\frac{\gamma-1}{2}M^2)} = -\frac{M^2(\gamma-1)}{(1+\frac{\gamma-1}{2}M^2)}\frac{dM}{M} \tag{20}$$

Now plug back in we have:

$$\frac{du}{u} = \frac{dM}{M} - \frac{\frac{\gamma-1}{2}M^2}{(1+\frac{\gamma-1}{2}M^2)}\frac{dM}{M} = \frac{1}{(1+\frac{\gamma-1}{2}M^2)}\frac{dM}{M} \tag{21}$$

Recall previously:

$$\frac{du}{u} = \frac{1}{\sqrt{M^2-1}}d\nu \tag{22}$$

Now we have:

$$\boxed{d\nu = \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}} \tag{23}$$

If we assume finite angle and do the integration, after some magic math:

$$\boxed{\nu_2-\nu_1 = [\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}]_{M_1}^{M_2}} \tag{24}$$

Notice that here:

$$\delta = \nu_2 - \nu_1 \tag{25}$$

Therefore, if given the **turning angle** and M_1, we can use this equation to get M_2. Notice that there is no analytical solution, we have to either use **iterative method** or find ν **as a function of M and tabulate or graph solution.**

3.2.1 Reference Condition Method

If we want to find $\nu = \nu(M)$ to get M_2, we need to **choose (arbitrary) reference condition.** For example, if we choose $\nu = 0$ at $M = 1$, then the expression becomes:

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1} \tag{26}$$

Here ν represents angle through which **a sonic flow would have to turn to reach M.**

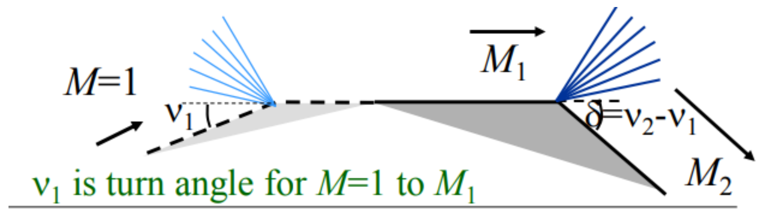

Figure 6: Reference Condition Method

3.2.2 Tabular Solution

Procedures:

1. Given M_1 and δ
2. Find ν_1 for given M_1 from table
3. Get ν_2 from $\delta = \nu_2 - \nu_1$
4. Check ν_2 in table to find M_2
5. Use **isentropic flow** relations to find T_2, p_2, since expansion is isentropic (no shock).

4 Prandtl Meyer Fan Angle

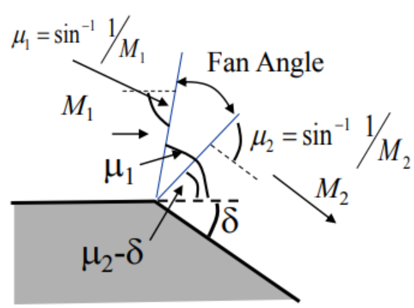

Figure 7: Fan Angle

The **Fan Angle** is defined as the angle between the **first and last** Mach wave. Using this we can know when expansion has ended in flow field for a given distance away from wall.

$$\text{Fan Angle} = \mu_1 - (\mu_2 - \delta) = (\mu_1 - \mu_2) + (\nu_2 - \nu_1) \tag{27}$$

5 Maximum Prandtl Meyer Angle

If we plot ν as a function of M:

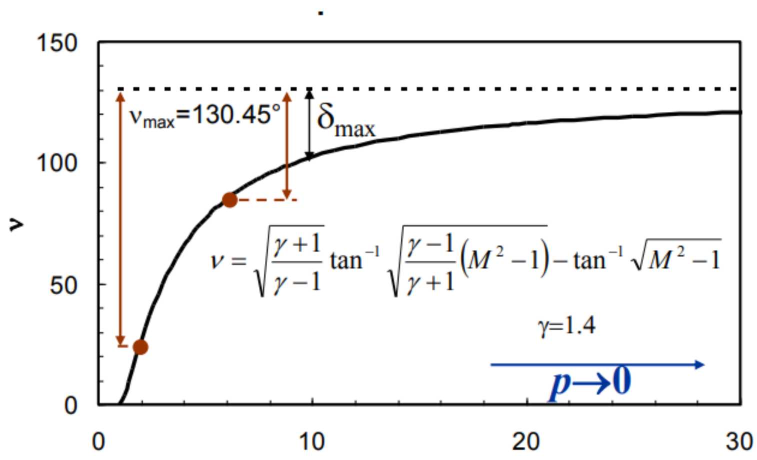

Figure 8: High Mach Number Cases

Some observations:

1. As M increases, the flow will reach **maximum Prandtl Meyer Angle** (when $\gamma = 1.4, \nu \approx 130.5^o$)
2. So as M increases, **maximum turn angle (δ_{max})** will decrease

Recall the equation:

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1} \tag{28}$$

Based on trigonometric function relationships:

$$M \to \infty: \ \tan^{-1}(\infty) = 90^o \tag{29}$$

Therefore we have:

$$\boxed{\nu_{max} = (\sqrt{\frac{\gamma+1}{\gamma-1}} - 1)90^o} \tag{30}$$

Therefore we can that if γ decreases (which means higher temperature or bigger molecules), **maximum Prandtl Meyer Angle** increases.

6 Reflected Expansion Wave

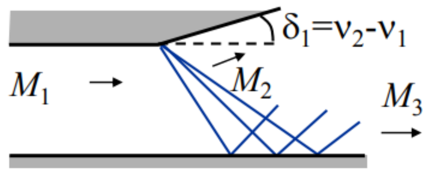

Figure 9: Reflected Expansion Wave

Considering a PM fan impinging on a flat wall. Then the incident expansion waves tend to turn flow away the lower wall. But vacuum could not be created, so flow must be turned back parallel to lower wall. The flow **opens up**, reflected waves are expansions.

If we want to get M_3:

$$\nu_3 = \delta_2 + \nu_2 = \delta_2 + (\delta_1 + \nu_1) = 2\delta + \nu_1 \tag{31}$$