Speed of Sound

1 Mach Number

The Mach number (often abbreviated as M) is a dimensionless quantity used in fluid dynamics to describe the speed of an object relative to the speed of sound in the surrounding medium (usually a fluid, such as air or water). Its expression is:

$$M = \frac{u}{a} \tag{1}$$

Where u is the flow speed, and a is the speed of sound. **But what is speed of sound?**

2 Speed of Sound

2.1 Derivation

The speed of sound refers to the speed at which **pressure waves**, such as sound waves, travel through a medium. The actual speed depends on the properties of the medium (such as air, water, or a solid material) and, particularly for gases, on temperature and pressure. Here, we want to derive the expressions.

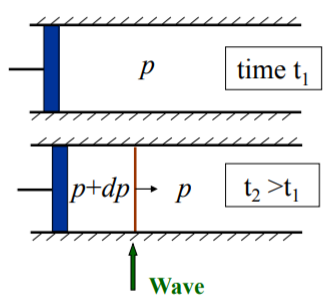

Figure 1: Pressure Wave

Consider the propagation of a **1D, adiabatic, weak** pressure wave traveling through **initially stationary simple compressible substance**. Now we want to know the wave speed c.

To do this, we first need to determine the reference frame:

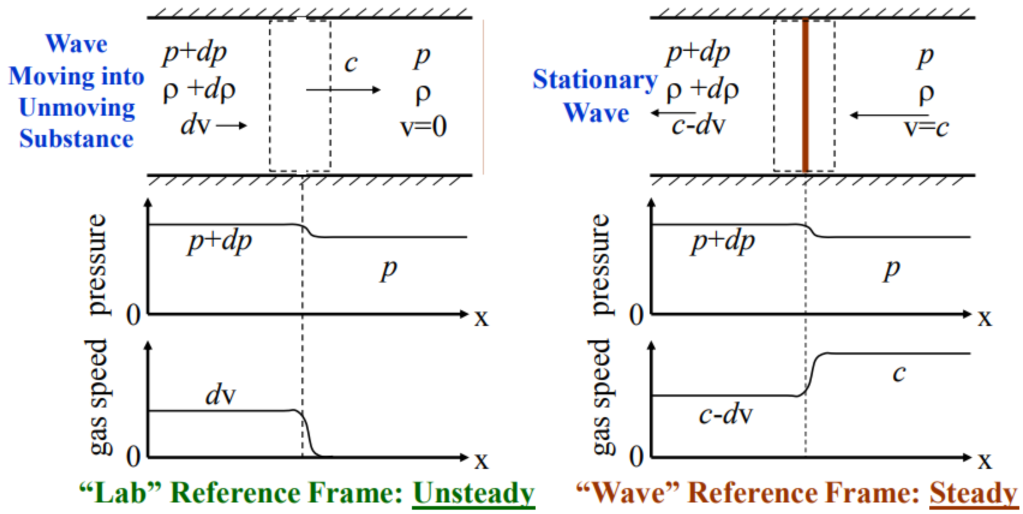

Figure 2: Reference Frame

To simplify the question, we choose the **steady wave reference frame**. The assumptions include:

1. Steady
2. Uniform
3. 1D
4. Inviscid
5. No body forces

We start from the **mass conservation:**

$$\dot{m}_{in} = \dot{m}_{out} = \dot{m} \tag{2}$$

$$\rho c A = (\rho + d\rho)(c - du)A \tag{3}$$

Rearrange we get:

$$du = c\frac{d\rho}{\rho + d\rho} \tag{4}$$

Then for the **momentum conservation**, the pressure difference causes the velocity difference:

$$A[(p + dp) - p] = \dot{m}[-(c - du) - (-c)] \tag{5}$$

$$Adp = (\rho c A)du \tag{6}$$

$$dp = \rho c du = \rho c^2 \frac{d\rho}{\rho + d\rho} \tag{7}$$

Finally we have:

$$c^2 = \frac{dp}{d\rho}(1 + \frac{d\rho}{\rho}) \tag{8}$$

In compressible flow, the terms ”weak wave” and ”strong wave” refer to the intensity of disturbances, such as shock waves or expansion waves, propagating through a medium. In details:

1. **Weak wave:** refers to waves where the changes in flow properties across the wave are small. Weak waves are generally seen in the initial stages or under conditions where the disturbances that caused the wave are relatively minor.
2. **Strong wave:** Strong waves are characterized by significant changes in flow properties across the wave. Non-linear effects become significant, which means that simple linear theories or approximations can no longer accurately predict the behavior across the wave.

There are some remarks for **very weak wave**:

1. **Wave speed = sound wave**
2. $d\rho/\rho << 1$
3. We already assumed reversible/inviscid, so if we **add adiabatic we get isentropic**

Therefore, we get:

$$dp = \frac{\partial p}{\partial \rho}|_s d\rho + \frac{\partial p}{\partial s}|_\rho \cancelto{0}{ds} \tag{9}$$

$$\frac{dp}{d\rho} = \frac{\partial p}{\partial \rho}|_s \tag{10}$$

Finally we have:

$$c^2 \approx \frac{dp}{d\rho} = \frac{\partial p}{\partial \rho}|_s \tag{11}$$

This equation is valid for all simple compressible substances. For incompressible substance, $d\rho \approx 0$ and $a \to \infty$, so it actually approves that **incompressible assumption is ideal case.**

2.2 Ideal Gases

Start from Gibbs equation:

$$Tds = du + pdv = c_v dT + pd(\frac{1}{\rho}) = c_v dT + p(-\frac{d\rho}{\rho^2}) \tag{12}$$

Also, we can write that:

$$Tds = dh - vdp = c_p dT - \frac{1}{\rho}dp \tag{13}$$

Therefore we have:

$$dT = \frac{p}{\rho^2 c_v} d\rho \tag{14}$$

And:

$$dT = \frac{1}{\rho c_p} dp \tag{15}$$

Rearrange:

$$\frac{p}{\rho^2 c_v} d\rho = \frac{1}{\rho c_p} dp \tag{16}$$

$$\frac{dp}{d\rho} = \frac{c_p}{c_v}\frac{p}{\rho} = \gamma \frac{p}{\rho} \tag{17}$$

Recall the ideal gas law, then we have **speed of sound for non-react TPG:**

$$a = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma RT} \tag{18}$$

Notice that if the molar weight is large, then value of R will be smaller ($R = \frac{\bar{R}}{MW}$), so the speed of sound would also be smaller. **Therefore, sound propagates faster in light gas.**