Boundary Layer Equations

1 Introduction

1.1 Wall-Bounded Flow

Wall-bounded flow refers to a fluid flow (usually of a gas or liquid) that is constrained or influenced by a boundary such as a wall or a surface. One key characteristic of wall-bounded flows is the velocity profile. This describes how the speed of the fluid changes from the wall to the interior of the flow. Another important factor in wall-bounded flows is the formation of a boundary layer.

1.2 Boundary Layer

Boundary layer is a thin layer of fluid at the wall where the effects of viscosity are important and where the fluid velocity changes rapidly from zero at the wall to the free stream value. This layer can be subject to separation under certain conditions, leading to phenomena like flow separation and stall in aerodynamics.

2 Derivation

2.1 Boundary Layer Assumption

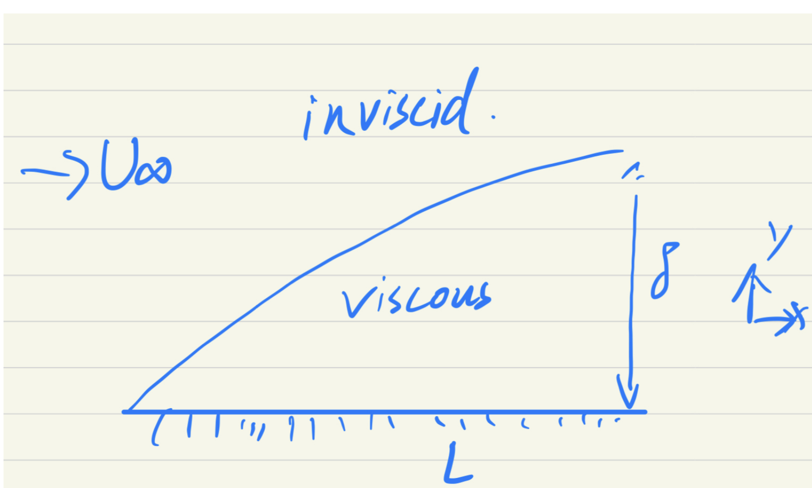

Figure 1: Boundary Layer Approximations.

1. Boundary Layer thickness δ much smaller than length L
2. Newtonian Fluid
3. Incompressible (Constant Density), no body force
4. 2D (Infinite Wide), $\frac{\partial}{\partial z} = 0$
5. Steady Flow

2.2 Governing Equations

2.2.1 Continuity

Based on previous parallel flow derivation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

2.2.2 Momentum

Original x direction momentum equation:

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x \partial x} + \frac{\partial^2 u}{\partial y \partial y}] \tag{2}$$

Based on the steady flow assumption:

$$\rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x \partial x} + \frac{\partial^2 u}{\partial y \partial y}] \tag{3}$$

Similarly, in y direction:

$$\rho(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \mu[\frac{\partial^2 v}{\partial x \partial x} + \frac{\partial^2 v}{\partial y \partial y}] \tag{4}$$

2.3 Order of Magnitude Analysis

2.3.1 Approximation

1. $\delta << L$
2. $\frac{\delta}{\delta x} \approx \frac{1}{L}$, $\frac{\delta}{\delta y} \approx \frac{1}{\delta}$
3. $\frac{\delta u}{\delta x} \approx \frac{U_\infty}{L}$, $\frac{\delta v}{\delta y} \approx \frac{v}{\delta}$
4. Based on continuity equation, $\frac{v}{\delta} \approx \frac{U_\infty}{L}$, $v \approx \frac{U_\infty \delta}{L}$

2.3.2 X Momentum OMA

$$[U_\infty \frac{U_\infty}{L}] + [\frac{U_\infty \delta}{L}\frac{U_\infty}{\delta}] \approx [?] + \nu[\cancelto{0}{\frac{U_\infty}{L^2}}, \frac{U_\infty}{\delta^2}] \tag{5}$$

$$[\frac{U_\infty^2}{L}] \approx [?] + [\nu \frac{U_\infty}{\delta^2}] \tag{6}$$

2.3.3 Y Momentum OMA

$$[U_\infty \frac{\frac{U_\infty \delta}{L}}{L}] + [\frac{U_\infty \delta}{L}\frac{\frac{U_\infty \delta}{L}}{\delta}] \approx [?] + \nu[\cancelto{0}{\frac{\frac{U_\infty \delta}{L}}{L^2}}, \frac{\frac{U_\infty \delta}{L}}{\delta^2}] \tag{7}$$

$$[\frac{U_\infty^2 \delta}{L^2}] \approx [?] + [\nu \frac{U_\infty}{\delta L}] \tag{8}$$

2.3.4 Combination

$$\delta << L \tag{9}$$

$$\frac{U_\infty^2 \delta}{L^2} << \frac{U_\infty^2}{L} \tag{10}$$

$$\nu \frac{U_\infty}{\delta L} << \nu \frac{U_\infty}{\delta^2} \tag{11}$$

Therefore, y momentum equation is negligible, and $\frac{\delta P}{\delta y}$ is negligible, so $P = P(x)$ only. At $y \approx \delta, u = U_\infty$:

$$U_\infty \frac{\partial U_\infty}{\partial x} + v \frac{\partial U_\infty}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu \frac{\partial^2 U_\infty}{\partial y \partial y} \tag{12}$$

We know that when y is large, U_∞ is not dependent on y, therefore:

$$U_\infty \frac{\partial U_\infty}{\partial x} = -\frac{1}{\rho}\frac{dP}{dx} \tag{13}$$

Integrate this, we can get the Bernoulli Equation:

$$P_\infty + \frac{1}{2}\rho U_\infty^2 = const \tag{14}$$

Plug in back to original equation, we can get the incompressible, 2D Boundary Layer Equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{15}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y \partial y} \tag{16}$$

Boundary conditions include:

1. No slip: $u = 0$ at $y = 0$
2. Impermeable: $v = 0$ at $y = 0$
3. Smooth Approach: $u \to U_\infty$ at $y \to \infty$