Boundary Layer Solutions

1 Important BL Parameters

1.1 99% BL Thickness

δ_{99} is the distance from wall such that $u = 0.99U_\infty$.

1.2 Displacement Thickness

Based on mass conservation, mass flow deficit due to $u < U_\infty$ in decelerated flow in boundary layer is:

$$\rho \int U_\infty \, dy - \rho \int u \, dy = \rho \int (U_\infty - u) \, dy \tag{1}$$

We define δ^* as:

$$\delta^* = \int_0^\infty (1 - \frac{u}{U_\infty}) \, dy \tag{2}$$

1.3 Momentum Thickness

Based on momentum conservation, reduced mass flow also implies less momentum.

$$\rho U_\infty^2 \theta = \int_0^\infty u(U_\infty - u) \, dy \tag{3}$$

$$\theta = \int_0^\infty \frac{u}{U_\infty}(1 - \frac{u}{U_\infty}) \, dy \tag{4}$$

1.4 Wall Shear Stress

For Newtonian fluid and impermeable wall ($v = 0$ for all x):

$$\tau_w = \mu \frac{\partial u}{\partial y}|_{y=0} \tag{5}$$

1.5 Friction Coefficient

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} \tag{6}$$

2 Boundary Layer Transformations (Simplified)

Assume a 2D stream function as:

$$\psi(x, y) = g(x) f(\eta) \tag{7}$$

η is to be non-dimensional, same as $f(\eta)$, which means $g(x)$ has the dimensions $[L^2][T^{-1}]$. Then, we define η as non-dimensional distance:

$$\eta = \frac{y}{\frac{g(x)}{U_\infty}} = \frac{U_\infty y}{g(x)} \tag{8}$$

Based on the definition of stream function:

$$u = \frac{\partial \psi}{\partial y} \tag{9}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{10}$$

Plug into equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{11}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U_\infty \frac{dU_\infty}{dx} + \nu \frac{\partial^3 \psi}{\partial y^3}$$

Then apply the transformation: e.g.

$$\frac{\partial \psi}{\partial y} = g(x)\frac{\partial f}{\partial y} = g(x) f' \frac{\partial \eta}{\partial y} = U_\infty f'$$

where primes are derivatives in η, and we have used $\partial \eta / \partial y = U_\infty / g$.
For derivatives in x, using the quotient rule of calculus, we get

$$\frac{\partial \eta}{\partial x} = y \frac{d}{dx}\left(\frac{U_\infty}{g}\right) = y \frac{g dU_\infty/dx - U_\infty dg/dx}{g^2}$$

With $\eta = U_\infty y / g$, to express stuff in terms of η, as far as possible:

$$\frac{\partial \eta}{\partial x} = \eta \left[\frac{1}{U_\infty}\frac{dU_\infty}{dx} - \frac{1}{g}\frac{dg}{dx}\right]$$

The math does get longer yet:

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial \psi}{\partial y}\right] = \frac{\partial}{\partial x}[U_\infty f'] = U_\infty f'' \frac{\partial \eta}{\partial x} + f' \frac{dU_\infty}{dx}$$

After some **magical math**, we get:

$$f''' + \frac{g}{\nu U_\infty}\frac{dg}{dx}ff'' + \frac{g^2}{\nu U_\infty^2}\frac{dU_\infty}{dx}(1 - f'^2) = 0 \tag{12}$$

Here, we define:

$$\alpha = \frac{g}{\nu U_\infty}\frac{dg}{dx} \tag{13}$$

$$\beta = \frac{g^2}{\nu U_\infty^2}\frac{dU_\infty}{dx} \tag{14}$$

To make sure f and its derivatives are all non-dimensional, both α and β should be non-dimensional constants. A different value of α simply changes the definition of $g(x)$ by a constant factor. Then, without loss of generality, we take $\alpha = 1$.

$$\frac{d}{dx}(\frac{g^2}{U_\infty}) = \frac{1}{U_\infty}\frac{dg^2}{dx} - \frac{g^2}{U_\infty^2}\frac{dU_\infty}{dx} = (2-\beta)\nu \tag{15}$$

Integrating in x, where x is measured from the leading edge, we get:

$$g^2 = (2-\beta)\nu U_\infty x \tag{16}$$

$$\beta = \frac{(2-\beta)\nu U_\infty x}{U_\infty^2 \nu}\frac{dU_\infty}{dx} \tag{17}$$

Or:

$$\frac{1}{U_\infty}\frac{dU_\infty}{dx} = (\frac{\beta}{2-\beta})(\frac{1}{x}) \tag{18}$$

Integration on both sides will give:

$$U_\infty(x) = Cx^m \tag{19}$$

Where C is a constant, and

$$m = \frac{\beta}{2-\beta} \tag{20}$$

$$\beta = \frac{2m}{m+1} \tag{21}$$

3 Blasius Solution

3.1 Derivation

Based on the power law, $m = 0$ means the boundary layer over a flat plate, with zero angle of attack. This also implies $\beta = 0$, then:

$$g = \sqrt{2\nu U_\infty x} \tag{22}$$

$$\eta = \frac{U_\infty y}{\sqrt{2\nu U_\infty x}} = \frac{y}{\sqrt{2\nu x/U_\infty}} \tag{23}$$

Finally it reduces to:

$$f^{'''} + ff^{''} = 0 \tag{24}$$

Check the Blasius table to get the solution.

3.2 Velocity

$$u = U_\infty f^{'} \tag{25}$$

$$v = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}(\eta f^{'} - f) \tag{26}$$

3.3 Thickness

$$\delta_{99} = 5\sqrt{\frac{\nu x}{U_\infty}} \tag{27}$$

$$\delta^* = \sqrt{\frac{\nu x}{U_\infty}}\int_0^\infty 1 - f^{'}\, d\eta = 1.7208\sqrt{\frac{\nu x}{U_\infty}} \tag{28}$$

$$\theta(x) = 0.664\sqrt{\frac{\nu x}{U_\infty}} \tag{29}$$

3.4 Wall Shear Stress and Friction Coefficient

$$\tau_w = \mu U_\infty f^{''}(0)\sqrt{\frac{\nu x}{U_\infty}} \tag{30}$$

$$C_f(x) = 2f^{''}(0)\sqrt{\frac{\nu}{U_\infty x}} = 0.664 {Re_x}^{-1/2} \tag{31}$$

4 Other Solutions

Power Law:

$$U_\infty = Ax^m \tag{32}$$

4.1 m = 0

Blasius Solution.

4.2 m = 1

Stagnation flow, but in the region far from the stagnation point.

4.3 m greater than 0

Accelerating boundary layer, with negative (favorable) pressure gradient.

$$\frac{1}{\rho}\frac{dP}{dx} = -U_\infty \frac{dU_\infty}{dx} = -A^2 x^m m x^{m-1} < 0 \tag{33}$$

4.4 m smaller than 0

Decelerating boundary layer, with positive (adverse) pressure gradient. Separation can occur.