Energy and Enthalpy Equations

1 Introduction

1.1 Energy

In the context of fluid dynamics, energy can be described in several forms: kinetic, potential, and internal energy. These forms are generally associated with the motion of the fluid, the position of the fluid, and the temperature and pressure of the fluid respectively. They all contribute to the total energy of a fluid system.

1. **Kinetic Energy:** This is associated with the motion of the fluid. For a small fluid element, the kinetic energy is given by $\frac{1}{2}\rho v^2$, where ρ is the fluid density and v is the fluid velocity.

2. **Potential Energy:** This is related to the position of the fluid in a gravitational field (or any other potential field). For a fluid element at height h above some reference point, the potential energy is given by $\rho g h$, where g is the acceleration due to gravity.

3. **Internal Energy:** This is associated with the temperature and pressure of the fluid, which account for the random motion and interaction of the individual molecules. It can include thermal energy, which is associated with the temperature of the fluid, and energy associated with the fluid's pressure.

1.2 Enthalpy

In fluid dynamics, enthalpy is a thermodynamic property of a fluid that is particularly important when analyzing flowing fluids, especially in the context of open systems like turbines, compressors, and nozzles. Enthalpy is a combined measure of the internal energy of a fluid and the work done by the fluid as it displaces its surroundings.

1.3 Material Volume

The volume occupied by the same body of fluid at all times.

2 Energy Equations

In this scenario, we ignore the potential energy, just focus on the internal and kinetic energy, then the total energy per mass could be expressed as:

$$e_{tot} = e + \frac{1}{2}\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{u}} \tag{1}$$

Based on the first thermodynamics law:

$$de_{tot} = \delta W + \delta Q \tag{2}$$

Integrate over material volume:

$$E_{tot} = \int_{V(t)} \rho(e + \frac{1}{2}\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{u}})\, dV \tag{3}$$

Take the time derivative to get the material derivative:

$$\frac{dE_{tot}}{dt} = \int_{V(t)} [\rho\frac{De}{Dt} + \rho\frac{D}{Dt}(\frac{1}{2}\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{u}})]\, dV \tag{4}$$

The work is caused by the forces, including body force:

$$\int_{V(t)} \rho(\underline{\boldsymbol{f}} \cdot \underline{\boldsymbol{u}})\, dV = \int_{V(t)} \rho u_i f_i\, dV \tag{5}$$

Also surface force (Based on the definition, we define the second letter as the force direction. Before we already use i as the force direction.):

$$\int_{A(t)} \tau_{ji} n_j u_i\, dA = \int_{V(t)} \frac{\partial}{\partial x_j}(\tau_{ji} u_i)\, dV \tag{6}$$

Q is the heat quantity, can be expressed using **Fourier's Law of Heat Conduction:**

$$\underline{\boldsymbol{q}} = -k\nabla T \tag{7}$$

Several important things:

1. $\underline{\boldsymbol{q}}$ here is the heat flux, with the unit as W/m^2. The unit of heat is Joule (J), the unit of power is Watt ($W = J/s$), flux is always per unit time per unit area, so $\dfrac{1}{s \cdot m^2}$
2. k here is the heat conductivity, with the unit as $W/(m \cdot K)$
3. ∇T is the gradient of the temperature, which is a vector with unit as $\frac{K}{m}$
4. Negative sign means that the heat flows down the temperature gradient.

For the surface, the unit vector $\underline{\boldsymbol{n}}$ always represent the outward direction, so the heat transfer into the system:

$$\int_{A(t)} -\underline{\boldsymbol{q}}\underline{\boldsymbol{n}}\, dA = \int_{V(t)} \nabla \cdot (k\nabla T)\, dV = \int_{V(t)} \frac{\partial}{\partial x_i}(k\frac{\partial T}{\partial x_i})\, dV \tag{8}$$

Therefore, we get the total energy equation:

$$\int_{V(t)} [\rho \frac{De}{Dt} + \rho \frac{D}{Dt}(\frac{1}{2}\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{u}})] \, dV = \int_{V(t)} [\rho u_i f_i + \frac{\partial}{\partial x_j}(\tau_{ji} u_i) + \frac{\partial}{\partial x_i}(k \frac{\partial T}{\partial x_i})] \, dV \tag{9}$$

$$\rho \frac{De}{Dt} + \rho \frac{D}{Dt}(\frac{1}{2} u_i u_i) = \rho u_i f_i + \frac{\partial}{\partial x_j}(\tau_{ji} u_i) + \frac{\partial}{\partial x_i}(k \frac{\partial T}{\partial x_i}) \tag{10}$$

The Kinetic energy equation could be obtained from NS equation:

$$\rho \frac{D u_i}{Dt} = \rho f_i + \frac{\partial \tau_{ji}}{\partial x_j} \tag{11}$$

$$\rho u_i \frac{D u_i}{Dt} = \rho u_i f_i + u_i \frac{\partial \tau_{ji}}{\partial x_j} \tag{12}$$

$$\rho \frac{D(\frac{1}{2} u_i u_i)}{Dt} = \rho u_i f_i + \frac{\partial}{\partial x_j}(\tau_{ji} u_i) - \tau_{ji} \frac{\partial u_i}{\partial x_j} \tag{13}$$

Then, we can get the internal energy equation:

$$\rho \frac{De}{Dt} = \rho \frac{De}{Dt} + \rho \frac{D}{Dt}(\frac{1}{2} u_i u_i) - \rho \frac{D}{Dt}(\frac{1}{2} u_i u_i) \tag{14}$$

$$\rho \frac{De}{Dt} = \frac{\partial}{\partial x_i}(k \frac{\partial T}{\partial x_i}) + \tau_{ji} \frac{\partial u_i}{\partial x_j} \tag{15}$$

Recall the definition of strain rate tensor and rotation rate tensor:

$$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = S_{ji} \tag{16}$$

$$R_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) = R_{ji} \tag{17}$$

Then:

$$\tau_{ji} \frac{\partial u_i}{\partial x_j} = \tau_{ji}(S_{ij} + R_{ij}) \tag{18}$$

Because τ_{ji} and S_{ij} are symmetric, R_{ij} is asymmetric:

$$\tau_{ji} \frac{\partial u_i}{\partial x_j} = \tau_{ji} S_{ij} = \tau_{ij} S_{ij} \tag{19}$$

$$\begin{aligned} \tau_{ij} S_{ij} &= S_{ij}[-p\delta_{ij} + 2\mu S_{ij} - \frac{2}{3}\mu \Delta \delta_{ij}] \\ &= -P S_{ii} + 2\mu S_{ij} S_{ij} - \frac{2}{3}\mu \Delta S_{ii} \\ &= -P\Delta + 2\mu(S_{ij} S_{ij} - \frac{1}{3}\Delta^2) \end{aligned} \tag{20}$$

If we define Φ as:

$$\Phi = 2(S_{ij}S_{ij} - \frac{1}{3}\Delta^2) \tag{21}$$

The internal energy for Newtonian fluid could also expressed as:

$$\rho\frac{De}{Dt} = -P\Delta + \mu\Phi + \frac{\partial}{\partial x_i}(k\frac{\partial T}{\partial x_i}) \tag{22}$$

$$\rho\frac{De}{Dt} = -P\Delta + \mu\Phi + \nabla \cdot (k\nabla T) \tag{23}$$

Some remarks:

1. $-P\Delta$ represents the work done by pressure
2. $\mu\Phi$ represents the generation of heat by viscosity
3. $\frac{\partial}{\partial x_i}(k\frac{\partial T}{\partial x_i})$ represents the heat transport by conduction

If we define d_{ij} as the deviatoric (anisotropic) part of S_{ij}, then:

$$S_{ij} = d_{ij} + \frac{1}{3}\Delta\delta_{ij} \tag{24}$$

$$d_{ii} = 0 \tag{25}$$

Therefore:

$$\begin{aligned} S_{ij}S_{ij} &= (d_{ij} + \frac{1}{3}\Delta\delta_{ij})(d_{ij} + \frac{1}{3}\Delta\delta_{ij}) \\ &= d_{ij}d_{ij} + \frac{2}{3}\Delta d_{ij}\delta_{ij} + \frac{1}{9}\Delta^2\delta_{ij}\delta_{ij} \\ &= d_{ij}d_{ij} + \frac{2}{3}\Delta d_{ii} + \frac{1}{9}\Delta^2 \cdot 3 \\ &= d_{ij}d_{ij} + \frac{1}{3}\Delta^2 \end{aligned} \tag{26}$$

Therefore, we can prove that

$$\Phi = 2d_{ij}d_{ij} \geq 0 \tag{27}$$

Notice that the term $\mu\Phi$ appears on the RHS of the internal energy equation, and $-\mu\Phi$ appears on the RHS of kinetic energy equation, meaning that flow is converting KE into heat by viscosity.

3 Enthalpy Equation

3.1 Derivation

Based on the definition of enthalpy:

$$h = e + \frac{P}{\rho} \tag{28}$$

Using chain rule:

$$dh = de + (\frac{1}{\rho})dP - (\frac{P}{\rho^2})d\rho \tag{29}$$

Transfer to the material derivative:

$$\rho\frac{Dh}{Dt} = \rho\frac{De}{Dt} + \frac{DP}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt} \tag{30}$$

Using the internal energy equation:

$$\begin{aligned} \rho\frac{Dh}{Dt} &= -P\Delta + \mu\Phi + \nabla\cdot(k\nabla T) + \frac{DP}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt} \\ &= -P(\nabla + \frac{1}{\rho}\frac{D\rho}{Dt}) + \mu\Phi + \nabla\cdot(k\nabla T) + \frac{DP}{Dt} \end{aligned} \tag{31}$$

From continuity:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{\boldsymbol{u}}) = 0 \tag{32}$$

$$\frac{\partial\rho}{\partial t} + \underline{\boldsymbol{u}}\cdot(\nabla\rho) + \rho(\nabla\cdot\underline{\boldsymbol{u}}) = 0 \tag{33}$$

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{1}{\rho}\underline{\boldsymbol{u}}\cdot(\nabla\rho) + (\nabla\cdot\underline{\boldsymbol{u}}) = 0 \tag{34}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\underline{\boldsymbol{u}} = 0 \tag{35}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \Delta = 0 \tag{36}$$

Finally, we get the Enthalpy equation:

$$\rho\frac{Dh}{Dt} = \mu\Phi + \nabla\cdot(k\nabla T) + \frac{DP}{Dt} \tag{37}$$

3.2 Order of Magnitude Analysis

Some basic dimensions:

1. $\rho_\infty, \mu, U_\infty, L$
2. Pressure difference: $\rho_\infty U_\infty^2$
3. Stagnation temperature difference: $(\Delta T)_0$
4. Time: L/U_∞

From thermodynamics:

$$dh = C_p dT \tag{38}$$

Therefore the **Convection** term:

$$\rho \frac{Dh}{Dt} \propto \frac{\rho_\infty C_p (\Delta T)_0}{L/U_\infty} \tag{39}$$

Compressibility term:

$$\frac{DP}{Dt} \propto \frac{\rho_\infty U_\infty^2}{L/U_\infty} \tag{40}$$

Conduction term:

$$\nabla \cdot (k \nabla T) \propto \frac{k(\Delta T)_0}{L^2} \tag{41}$$

Viscous Dissipation term:

$$\mu\Phi = \mu(2d_{ij}d_{ij}) \propto \mu \frac{U_\infty^2}{L^2} \tag{42}$$

3.3 Non-dimensional Parameters

Prandtl Number is defined as the ratio of momentum to thermal diffusivities:

$$Pr = \frac{\nu}{\frac{k}{\rho C_p}} = \frac{\mu C_p}{k} \tag{43}$$

Reynolds Number is defined as the ratio of momentum forces to viscous forces:

$$Re = \frac{\rho_\infty U_\infty L}{\mu} \tag{44}$$

From previous section:

$$\frac{Conduction}{Convection} = \frac{\frac{k(\Delta T)_0}{L^2}}{\frac{\rho_\infty C_p (\Delta T)_0}{L/U_\infty}} = \frac{k}{\rho_\infty U_\infty C_p L} \tag{45}$$

Also:

$$\frac{1}{Pr}\frac{1}{Re} = \frac{k}{\mu C_p}\frac{\mu}{\rho_\infty U_\infty L} = \frac{k}{\rho_\infty U_\infty C_p L} \tag{46}$$

Therefore:

$$\frac{1}{Pr}\frac{1}{Re} = \frac{Conduction}{Convection} \tag{47}$$