Hydrodynamic Stability

1 Overview

1.1 Definition

Hydrodynamic stability in fluid mechanics refers to the study of fluid flows that are in equilibrium, and what happens when they are subject to disturbances. This is important because even a small disturbance can lead to significant changes in the flow, such as transition from laminar to turbulent flow, or the onset of wave phenomena.

1.2 Types of Stability

Main idea is to introduce some types of disturbances into a steady laminar flow and see how the disturbance behave:

1. **Unstable:** if it grows or amplifies
2. **Stable:** if it decays or is attenuated
3. **Neutrally Stable:** if it remains at constant amplitude

2 Linear, Small Disturbance Theory

First we assume a simple base flow, such as a parallel pure shear flow:

$$U = U(y), V = W = 0, P = P(x, y) \tag{1}$$

A flow is unstable if it is unstable to "any" form of disturbance. It is proven that 2D disturbances are more de-stablizing than 3D disturbances, so it is appropriate to restrict analysis to 2D disturbances.

Introduce a 2D disturbance in the form of:

$$u' = u'(x, y, t), v' = v'(x, y, t), p' = p'(x, y, t) \tag{2}$$

So the resultant flow is given by:

$$u = U + u', v = v', w = 0, p = P + p' \tag{3}$$

Recall the X momentum equation:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu(\nabla^2 U) \tag{4}$$

Add disturbance:

$$\frac{\partial(U+u')}{\partial t} + (U+u')\frac{\partial(U+u')}{\partial x} + (V+v')\frac{\partial(U+u')}{\partial y} = -\frac{1}{\rho}\frac{\partial(P+p')}{\partial x} + \nu(\nabla^2(U+u')) \tag{5}$$

Rules for simplification:

1. Inside $\partial/\partial x$ and Inside $\partial/\partial y$, $U + u'$ could not be approximated to U
2. Outside, $U + u'$ could be approximated to U
3. $v'\partial u'/\partial y$ is too small, could be ignored

Therefore, we get:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu(\nabla^2(U+u')) \tag{6}$$

From base flow, assume steady flow:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{7}$$

Because $V = 0$, $\partial U/\partial x = 0$. Therefore,

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu(\nabla^2 U) \tag{8}$$

So we have:

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = \nu(\nabla^2 u') \tag{9}$$

Recall Y momentum equation:

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu(\nabla^2 V) \tag{10}$$

Add the disturbance:

$$\frac{\partial(V+v')}{\partial t} + (U+u')\frac{\partial(V+v')}{\partial x} + (V+v')\frac{\partial(V+v')}{\partial y} = -\frac{1}{\rho}\frac{\partial(P+p')}{\partial y} + \nu(\nabla^2(V+v')) \tag{11}$$

Similar with X momentum, simplify (again, $v'\partial v'/\partial y$ is too small, could be ignored):

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{1}{\rho}\frac{\partial(P+p')}{\partial y} + \nu(\nabla^2 v') \tag{12}$$

From base flow,

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} \tag{13}$$

Therefore:

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial y} = \nu(\nabla^2 v') \tag{14}$$

3 Orr-Sommerfeld Equations

3.1 Periodic 2D disturbances

2D disturbance itself satisfies continuity, so:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{15}$$

We can also use a periodic stream function to express the disturbance:

$$\psi(x,y,t) = \phi(y)exp[i(\alpha x - \beta t)] \tag{16}$$

Here:

1. $i = \sqrt{-1}$ allows us to consider disturbances that may potentially oscillate or propagate in addition to being amplified or attenuated.
2. α is real, means periodic in x, with wavelength $\lambda = 2\pi/\alpha$

If we assume that a general 2D disturbance can be decomposed into a sum of elementary disturbances each with a different wavelength; response of flow to disturbance can depend on the wavelength:

$$\beta = \beta_r + i\beta_i \tag{17}$$

$$\psi(x,y,t) = \phi(y)\exp[i(\alpha x - \beta_r t)]\exp(\beta_i t) \tag{18}$$

The sign of β_i controls the amplitude of the disturbance, unstable if $\beta_i > 0$

3.2 Derivation

From the disturbance stream function above:

$$u' = \frac{\partial \psi}{\partial y} = \phi'(y)\exp[i(\alpha x - \beta t)] \tag{19}$$

$$v' = -\frac{\partial \psi}{\partial x} = -i\alpha\phi(y)\exp[i(\alpha x - \beta t)] \tag{20}$$

Plug in back to the simplified disturbance equations, eliminate the pressure gradient terms by taking the curl, we can get the OS Equation:

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = -\frac{i}{\alpha Re}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \tag{21}$$

Here:

1. $Re = \frac{U_m\delta}{\nu}$
2. c is complex valued, $c = \frac{\beta}{\alpha} = c_R + ic_I$
3. $1/\alpha$ is a wavelength, while β has the dimension of frequency, c_R/α can be interpreted as the propagating speed of the disturbance
4. Amplification versus attenuation depends on whether $c_I > 0$ or < 0. $c_I = 0$ gives condition of neutral stability
5. Boundary conditions: disturbance must satisfy no-slip and impermeability at the walls, and also vanish in the freestream. $u' = 0 = v' \rightarrow \phi = 0 = \phi'$ at both $y = 0, y \rightarrow \infty$

3.3 Inviscid Instability

In the limit of $\nu \rightarrow 0$ or $Re \rightarrow \infty$, we can get the Rayleigh Equation:

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = 0 \tag{22}$$

Necessary conditions for instability:

1. Flow should have a point of inflexion ($U'(y) \neq 0$ but $U''(y) = 0$)
2. If a PI exists, it is further necessary that $U''(y)(U - U_{PI}) < 0$ somewhere in the velocity profile

3.4 Neutral Stability Curves

Apply O-S equation for a boundary layer, the locus of $c_I = 0$ in the $\alpha\delta - Re$ plane gives the neutral stability curves.

Observations:

1. Below certain Re_δ, disturbances of all wavelengths are attenuated. Above this number, some disturbances may grow.
2. Typically, Re_crit is lower for velocity profiles with a point of inflexion, because such flows may undergo inviscid instability
3. Critical Re for instability is much lower than critical Re for transition to turbulence, because the disturbances need to grow in amplitude before they become large enough for turbulence to develop
4. Viscosity may have some stabilizing effect for the flow (larger stable region)

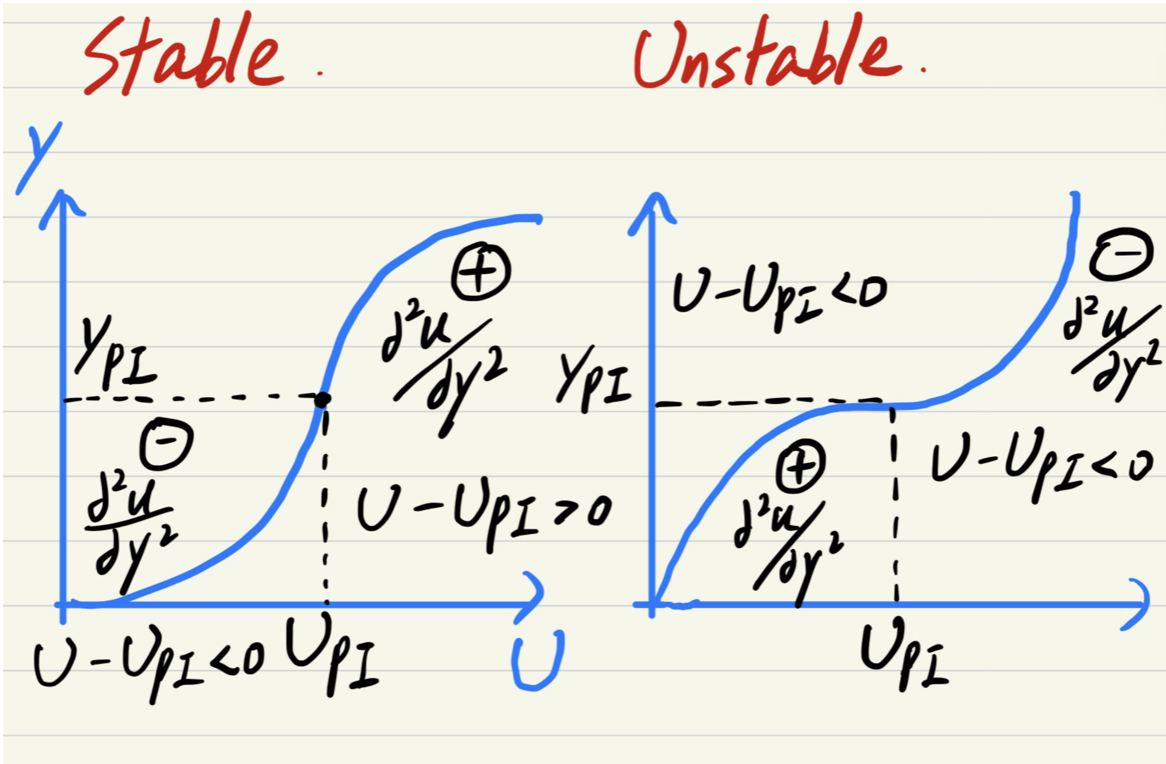
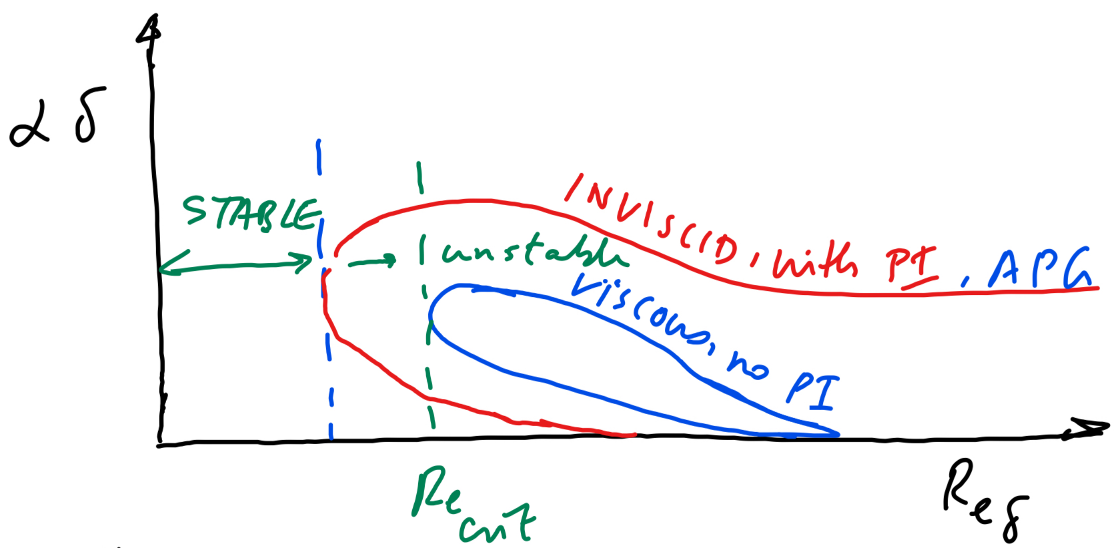


Figure 1: Point of Inflexion.



Observations.

Figure 2: Neutral Stability Curves.