Stokes Flow

1 Introduction

A common but not flawless interpretation of Re is as the ratio between *typical magnitudes* of inertial forces to viscous forces. In most engineering application, $Re >> 1$, but there are also some cases with $Re << 1$, such as duct of slowly-varying cross section (human digestive system), hydrodynamic lubrication (between bodies in relative motion), blood and so on.

2 Formula

For the constant density flow, if we ignore body force, we have:

$$\rho(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{1}$$

Notice that $\frac{\partial u_i}{\partial t}$ is called the **unsteady term**, as it is changing with time. And $u_j \frac{\partial u_i}{\partial x_j}$ is called the **non-linear** or **inertia term**. The term with the product of $u \times u$ is called non-linear. Inertia comes from Newton's second law $\underline{\boldsymbol{F}} = m\underline{\boldsymbol{a}}$. Integration of ρu will be the mass, and $\frac{\partial u_i}{\partial x_j}$ refers to the acceleration.

As mentioned in introduction, Reynolds number can be interpreted as the ratio between *typical magnitudes* of inertial forces to viscous forces. Therefore,

$$\frac{\rho \underline{\boldsymbol{u}} \cdot \nabla \underline{\boldsymbol{u}}}{\mu \nabla^2 \underline{\boldsymbol{u}}} \sim \frac{UL\rho}{\mu} = Re \tag{2}$$

Normally for the stokes flow, $Re << 1$, so we can ignore the nonlinear terms. The equation becomes:

$$\nabla p = \mu \nabla^2 \underline{\boldsymbol{u}} \tag{3}$$

From this equation, we can get many other relations. Since curl of the gradient of a scalar is always equal to 0, then:

$$\nabla \times \nabla p = \mu \nabla^2 (\nabla \times \underline{\boldsymbol{u}}) \tag{4}$$

$$0 = \mu \nabla^2 \underline{\boldsymbol{\omega}} \tag{5}$$

Also, we can take the divergence:

$$\nabla \cdot (\nabla p) = \mu \nabla^2 (\nabla \cdot \underline{\boldsymbol{u}}) \tag{6}$$

At the constant density situation:

$$\nabla^2 p = 0 \tag{7}$$

In 2D, based on the definition of stream function, we have:

$$u = \frac{\partial \psi}{\partial y} \tag{8}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{9}$$

Vorticity:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{10}$$

$$\omega = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} \tag{11}$$

$$\nabla^2 \psi = -\omega \tag{12}$$

From previous equation, we know:

$$\nabla^2 \omega = 0 \tag{13}$$

$$-\nabla^2 (\nabla^2 \psi) = 0 \tag{14}$$

$$\nabla^4 \psi = 0 \tag{15}$$

Notice that:

$$\nabla^4 = \frac{\partial^2}{\partial x^2}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + \frac{\partial^2}{\partial y^2}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \tag{16}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{17}$$