Thermal Boundary Layer

1 Overview

The thermal boundary layer in a fluid flow refers to the layer of fluid in the immediate vicinity of a bounding surface where the temperature varies between that at the surface and the temperature far from the surface. Several characteristics of themal boundary layer (assume hot surface):

1. At the hot surface itself, the air will be at the same temperature as the surface (due to the no-slip condition and the heat conduction). This is the start of the thermal boundary layer.

2. As we move further away from the surface, the air temperature will gradually decrease, transitioning from the surface temperature to the "free stream" temperature, which is the temperature of the air far from the surface. This change happens within the thermal boundary layer.

3. At the edge of the thermal boundary layer, the temperature has essentially reached the free stream temperature.

2 Governing Equations

2.1 Derivation

Starting from enthalpy equation:

$$\rho \frac{Dh}{Dt} = \mu\Phi + \nabla \cdot (k\nabla T) + \frac{DP}{Dt} \tag{1}$$

Now we assume the flow is steady and 2D:

$$\rho C_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}) = (u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y}) + k(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + 2\mu(S_{ij}S_{ij} - \frac{1}{3}\Delta^2) \tag{2}$$

Recall that:

$$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{3}$$

$$\Delta = \frac{\partial u_i}{\partial x_i} \tag{4}$$

Therefore in 2D:

$$S_{ij}S_{ij} = S_{11}^2 + S_{12}^2 + S_{21}^2 + S_{22}^2 \tag{5}$$

$$S_{11} = \frac{1}{2}(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}) = \frac{\partial u}{\partial x} \tag{6}$$

$$S_{22} = \frac{1}{2}(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}) = \frac{\partial v}{\partial y} \tag{7}$$

$$S_{12} = S_{21} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \tag{8}$$

$$2\mu(S_{ij}S_{ij} - \frac{1}{3}\Delta^2) = 2\mu[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2 - \frac{1}{3}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2] \tag{9}$$

Some assumptions of thermal boundary layer:

1. $\partial P/\partial y$ is negligible, same as hydrodynamic boundary layer.
2. $|\partial^2 T/\partial y^2| >> |\partial^2 T/\partial x^2|$, thermal boundary layer is also very thin.
3. $\partial u/\partial y$ much larger than other velocity gradients.

Therefore:

$$\rho C_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}) = u\frac{\partial P}{\partial x} + k\frac{\partial^2 T}{\partial y^2} + \mu(\frac{\partial u}{\partial y})^2 \tag{10}$$

Define δ_T as the thickness of the thermal boundary layer. Notice that for the terms include T we need to use δ_T, for the terms include velocity we need to use δ.

2.2 Order of Magnitude Analysis

Only consider the terms including Temperature:

$$[\rho C_p U_\infty \frac{(\Delta T)_0}{L}] + [\rho C_p U_\infty \frac{\delta}{L}\frac{(\Delta T)_0}{\delta_T}] \propto k\frac{(\Delta T)_0}{\delta_T^2} \tag{11}$$

2.2.1 If $\delta < \delta_T$

$$[\rho C_p U_\infty \frac{(\Delta T)_0}{L}] \propto k\frac{(\Delta T)_0}{\delta_T^2} \tag{12}$$

Therefore:

$$\delta_T^2 \propto \frac{kL}{\rho C_p U_\infty} \tag{13}$$

$$(\frac{\delta_T}{L})^2 \propto \frac{k}{\rho C_p U_\infty L} = \frac{1}{Pr}\frac{1}{Re} \tag{14}$$

For the laminar boundary layer:

$$\frac{\delta}{L} \propto Re_L^{-\frac{1}{2}} = (\frac{U_\infty L}{\nu})^{-\frac{1}{2}} = U_\infty^{-\frac{1}{2}} L^{-\frac{1}{2}} \nu^{\frac{1}{2}} \tag{15}$$

$$\delta \propto U_\infty^{-\frac{1}{2}} L^{\frac{1}{2}} \nu^{\frac{1}{2}} \tag{16}$$

Also:

$$Pr = \frac{\mu C_p}{k} = \rho\nu\ C_p k^{-1} \tag{17}$$

$$\delta_T \propto (\frac{kL}{\rho C_p U_\infty})^{\frac{1}{2}} = (kL)^{\frac{1}{2}} (\rho C_p U_\infty)^{-\frac{1}{2}} \tag{18}$$

Therefore:

$$\frac{\delta_T}{\delta} \propto \frac{(kL)^{\frac{1}{2}}(\rho C_p U_\infty)^{-\frac{1}{2}}}{U_\infty^{-\frac{1}{2}} L^{\frac{1}{2}} \nu^{\frac{1}{2}}} = k^{\frac{1}{2}} (\rho C_p \nu)^{-\frac{1}{2}} = Pr^{-\frac{1}{2}} \tag{19}$$

2.2.2 If $\delta > \delta_T$

$$[\rho C_p U_\infty \frac{\delta}{L} \frac{(\Delta T)_0}{\delta_T}] \propto k \frac{(\Delta T)_0}{\delta_T^2} \tag{20}$$

Rearrange:

$$\delta_T \propto \frac{kL}{\rho C_p U_\infty \delta} \tag{21}$$

$$\frac{\delta_T}{\delta} \propto \frac{kL}{\rho C_p U_\infty \delta^2} = \frac{k}{\rho C_p U_\infty} \frac{L}{\frac{\nu L}{U_\infty}} = \frac{1}{Pr} \tag{22}$$

2.2.3 Summary

$$\frac{\delta_T}{\delta} \propto : \begin{cases} Pr^{-1/2} & \text{if } \delta < \delta_T \\ Pr^{-1} & \text{if } \delta > \delta_T \end{cases}$$

For most gases, $Pr <= 1$ (0.72 for air under standard conditions). For most liquids, $Pr > 1$. If $Pr < 1$, means heat by conduction is **more efficient** than diffusion of momentum by viscosity: expect effect of heated/ cooled wall to reach further into the fluid, so $\delta_T > \delta$.