Vorticity

1 Definition

At its most fundamental level, vorticity is the measure of the rotation or spinning motion in a fluid. The concept is similar to angular velocity in solid body rotation, but it's specifically applied to fluid dynamics. Consider a small parcel of fluid in a flow; vorticity describes how this parcel would rotate if you were to follow it along its path. **If the fluid is rotating clockwise, the vorticity is negative, and if it's rotating counterclockwise, the vorticity is positive.** The direction of the vorticity vector indicates the axis around which the rotation happens. This axis is perpendicular to the plane in which the rotation occurs, following the right-hand rule (**if the fingers of the right hand curl from the rotation direction, the thumb points to the vorticity direction**). Vorticity is crucial in the study of weather patterns, ocean currents, and the flow of air around airplane wings, among other applications. For instance, tornadoes and cyclones are characterized by intense vorticity, and the vorticity of air flow around an airplane wing is a major factor in the lift the wing generates.

2 Math

2.1 Relations with Velocity

Recall the distortion of moving fluid element. In the picture, the fluid element is 2D. So the average rotation about the z axis is:

$$d\Omega_z = \frac{1}{2}(d\alpha - d\beta) \tag{1}$$

Both $d\alpha$ and $d\beta$ are directly related to velocity derivatives through the calculus limit:

$$d\alpha = \lim_{dt \to 0}(tan^{-1}\frac{\frac{\partial v}{\partial x}dxdt}{dx + \frac{\partial u}{\partial x}dxdt}) = \frac{\partial v}{\partial x}dt \tag{2}$$

$$d\beta = \lim_{dt \to 0}(tan^{-1}\frac{\frac{\partial u}{\partial y}dydt}{dy + \frac{\partial v}{\partial y}dydt}) = \frac{\partial u}{\partial y}dt \tag{3}$$

Therefore, the rate of rotation about the z axis is:

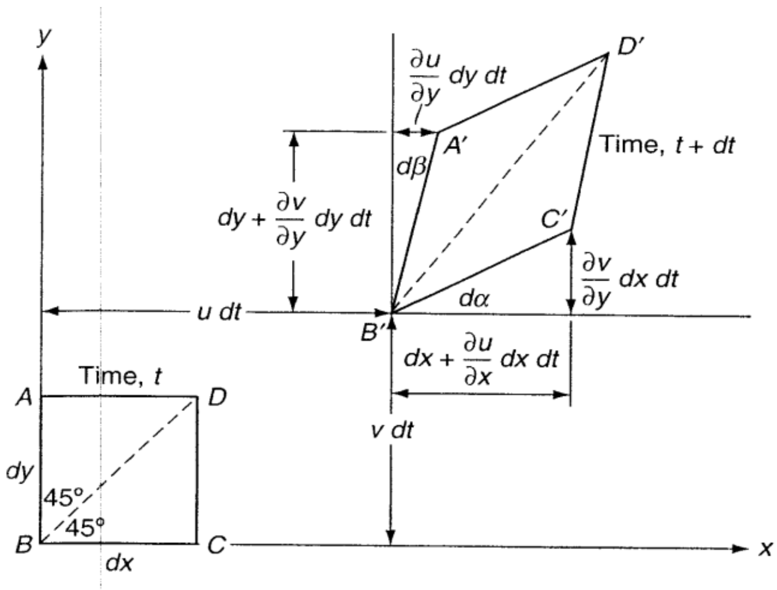

Figure 1: Distortion of Fluid Element[1].

$$\frac{d\Omega_z}{dt} = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \tag{4}$$

Similarly we could get:

$$\frac{d\Omega_x}{dt} = \frac{1}{2}(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) \tag{5}$$

$$\frac{d\Omega_y}{dt} = \frac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) \tag{6}$$

Therefore, we define the **vorticity**:

$$\underline{\boldsymbol{\omega}} = 2\frac{d\underline{\boldsymbol{\Omega}}}{dt} \tag{7}$$

or the curl of the velocity field:

$$\underline{\boldsymbol{\omega}} = \nabla \times \underline{\boldsymbol{u}} \tag{8}$$

so it has the characteristics:

$$\nabla \cdot \underline{\boldsymbol{\omega}} = \nabla \cdot (\nabla \times \underline{\boldsymbol{u}}) = 0 \tag{9}$$

which means the vorticity vector is **solenoidal**, or its divergence is zero everywhere. In other words, the field's "outflow" at every point in space is zero, suggesting that the field lines are closed loops or have no beginning or end. This property is a local one, meaning it applies in the vicinity of every point in the field.

The flow is **irrotational** if:

$$\underline{\boldsymbol{\omega}} = 0 \tag{10}$$

2.2 Transport Equation

Taking the curl of the momentum equation can gain the vorticity transport equation:

$$\frac{D\underline{\boldsymbol{\omega}}}{Dt} = \underline{\boldsymbol{\omega}} \cdot \nabla \underline{\boldsymbol{u}} + \nu \nabla^2 \underline{\boldsymbol{\omega}} \tag{11}$$

The two terms on the right hand side are **Vortex Stretching** and **Viscous Diffusion** terms. If the flow is inviscid, and $\underline{\boldsymbol{\omega}} = 0$ initially, it will remain zero at all times. In viscous flow, vorticity is oftern generated at solid boundaries.

References

[1] F. M. White. *Viscous Fluid Flow*. McGraw-Hill, 2006.