Wake and Mixing Layer

1 Wake

1.1 Overview

In fluid dynamics, a "wake" refers to the region of disturbed flow (often turbulent) downstream of a solid body moving through a fluid (liquid or gas). The wake is caused by the body obstructing the fluid flow. This obstruction creates a trail of turbulence, decreased pressure, and fluid velocity variation.

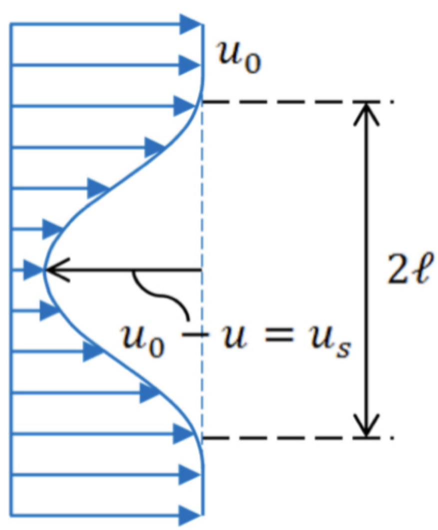

Figure 1: Wake.

1.2 Boundary Conditions

1. At $y = 0$, $u(x, y)$ is minimum
2. At $y = 0$, $\dfrac{\partial u}{\partial y} = 0, v = 0$ by symmetric
3. At $y = \pm\infty, u = U_\infty = const$

1.3 Governing Equations

Start from general free shear flow equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y \partial y} \tag{2}$$

Define $u_1(x, y)$ as the velocity deficit, then:

$$u_1(x, y) = U_\infty - u(x.y) \tag{3}$$

$$(U_\infty - u_1)\frac{\partial u_1}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y \partial y} \tag{4}$$

Let U_S denote max velocity deficit (at centerline): this is also a measure of change of velocity inside the flow.

At some long distance (ℓ) downstream: may expect $U_\infty - u_1 \approx U_\infty$. Estimate O.M of terms in equation above: respectively

$$\frac{U_\infty U_S}{\ell} \ , \quad \frac{U_S\delta}{\ell}\,\frac{U_S}{\delta} \ , \quad \nu\frac{U_S}{\delta^2}$$

where U_S decreases with x, while δ increases with x. But since $U_\infty \gg U_S$, the first convective term is much larger than the second. Hence equation simplifies to

$$U_\infty\frac{\partial u_1}{\partial x} = \nu\frac{\partial^2 u_1}{\partial y^2}$$

Looks like equation for "Stokes first problem" before, if we define $t = x/U_\infty$. But BC's different, so solution is different.

Similarity Solution

Assume a similarity solution for velocity defect, of the form

$$\frac{u_1(x,y)}{U_S(x)} = \text{fn. of } \frac{y}{\delta(x)}$$

Here δ is a diffusive length scale. By dimensional arguments

$$\delta(x) = 2\sqrt{\nu t} = 2\sqrt{\nu x/U_\infty}$$

(factor of 2 being for mathematical convenience later)

Momentum conservation

The momentum deficit in the wake per unit span is

$$F = \rho \int_{-\infty}^{\infty} u(U_\infty - u)\ dy \approx \rho U_\infty \int_{-\infty}^{\infty} u_1\ dy$$

"No external forces" implies F is independent of x. Hence

$$U_S(x)\delta(x) \propto x^0$$

$$\text{i.e.} \quad U_s(x)/U_\infty = Bx^{-1/2}$$

where B (of dimensions $[L]^{1/2}$) is to be determined.

Form of the velocity profile

$$u_1 = \frac{BU_\infty}{\sqrt{x}} g(\eta) \ \ ; \ \ \text{where} \ \ \eta = \frac{y}{2\sqrt{\nu x/U_\infty}}$$

Use chain rule to transform the velocity deficit equation:

$$g'' + 2(\eta g' + g) = 0$$

$$\Rightarrow \quad g'' + (2\eta g)' = 0$$

Integrating, using BC at $\eta = 0$

$$g' = -2\eta g \Rightarrow g = \exp\left(-\eta^2\right)$$

$$\text{i.e.} \quad u_1 = \frac{BU_\infty}{\sqrt{x}} \exp\left(\frac{-y^2}{4\nu x/U_\infty}\right)$$

Use integral constraint to determine B:

In our far-field approximation,

$$\begin{aligned}
F &= \rho U_\infty \int_{-\infty}^{\infty} u_1 \, dy \\
&= \rho \frac{BU_\infty^2}{\sqrt{x}} \int_{-\infty}^{\infty} \exp\left(\frac{-y^2}{4\nu x/U_\infty}\right) \, dy \\
&= \rho \frac{BU_\infty^2}{\sqrt{x}} \int_{-\infty}^{\infty} \exp\left(-\eta^2\right) \, d\eta \left(2\sqrt{\frac{\nu x}{U_\infty}}\right) \\
&= 2\rho BU_\infty^2 \sqrt{\tfrac{\nu}{U_\infty}} \tfrac{\sqrt{\pi}}{2} \Big[\operatorname{erf}(\infty) - \operatorname{erf}(-\infty) \Big] \\
&= 2\rho U_\infty^2 B \sqrt{\frac{\pi\nu}{U_\infty}}
\end{aligned}$$

Here F is the drag force *per unit span*. For a body of length L the drag coefficient (C_d, in 2D flow) is defined such that

$$F = C_d \, \tfrac{1}{2} \rho U_\infty^2 L$$

Equating these two formulas for F gives

$$2B\sqrt{\pi\nu/U_\infty} = C_d \, \tfrac{1}{2} L$$

We can write B in terms of the drag coefficient and Reynolds number

$$\frac{B}{\sqrt{L}} = \frac{C_d}{4\sqrt{\pi}} Re_L^{1/2}$$

With B expressed in terms of L, C_d and Re_L per the last equation on the last page, we can substitute in the expression $u_1 = ...$
It is best to re-arrange in a non-dimensional form:

$$\frac{u_1}{U_\infty} = \frac{C_d}{4\sqrt{\pi}} Re_L^{1/2} \left(\frac{x}{L}\right)^{-1/2} \exp\left(\frac{-y^2}{4\nu x/U_\infty}\right)$$

We recover U_S at $y = 0$. We have 3 basic dimensions (M,L,T) and 8 variables: u_1, U_∞, x, y, L, ν, F, ρ. The Buckingham-π theorem says to have 8-3=5 non-dimensional groups:

$$u_1/U_\infty,\ C_d,\ Re_L,\ x/L,\ \eta$$

The velocity (defect) profile is "bell-shaped" (sometimes called Gaussian). This formula has a singularity at $x = 0$: not a concern though, since have assumed "far downstream".

Additional Physical Comments

1. Higher viscosity implies lower Re_L overall: less velocity defect since viscous diffusion is more complete.

2. Half width: value of y where u_1/U_S is 0.5. i.e. exp(...) = 0.5.

3. Reynolds number $U_S(x)\delta(x)/\nu = const$. No tendency for transition to turbulence downstream.

Wake behind immersed flat plate

If Blasius solution applies:

$$C_d = 2(1.328)Re_L^{-1/2}$$

(Factor of 2 applied since plate is wetted on both sides.)

2 Mixing Layer

2.1 Overview

In fluid dynamics, the mixing layer is a type of shear flow which occurs when two parallel streams of fluid flow at different velocities. The difference in velocities at the two layers generates a shear stress and, through viscous effects, triggers instabilities that cause the two layers to mix together. This process is important in a variety of physical phenomena, from meteorological patterns in the atmosphere to flow in rivers and oceans, to the spread of pollutants, and in industrial applications.

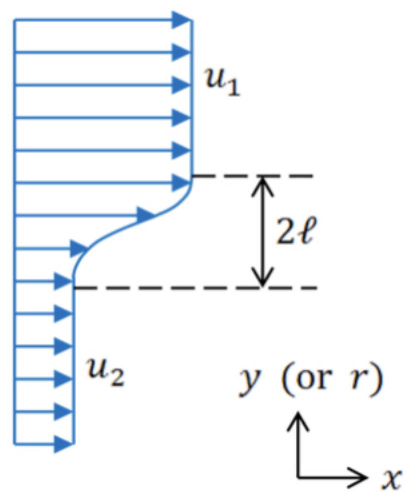

Figure 2: Mixing Layers.

2.2 Boundary Conditions

1. $y \to +\infty : u \to u_1, v \to 0$

2. $y \to +\infty : u \to u_2, v \to 0$

3.
$$y = 0 : \begin{cases} u(0^+) = u(0^-) \\ v(0^+) = v(0^-) \\ \mu \frac{\partial u}{\partial y}|_{0^+} = \mu \frac{\partial u}{\partial y}|_{0^-} \end{cases}$$