

Continuum Model

1 Introduction

Continuum model is a model in which variables are considered to vary **continuously over space and/or time**, as opposed to being discretized or represented only at specific points. Continuum models are typically used when dealing with systems that are well-described by continuous variables, such as temperature in a fluid or stress in a solid, rather than by discrete events or entities.

2 Derivatives

Recall the definition of the derivative:

$$\frac{df}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \quad (1)$$

It represents the instantaneous rate of change.

3 Case Study

3.1 Population Prediction

Assume the **population** is expressed as f , the **birth rate** is expressed as r , and the **death rate** is expressed as s . Then we have:

$$\frac{df}{dt} = rf - sf = (r - s)f \quad (2)$$

If the initial value is:

$$f(0) = c \quad (3)$$

Then we have the analytical solution:

$$f(t) = c \cdot \exp((r - s)t) \quad (4)$$

3.2 Population Cap

This time, suppose the environment only has resources to sustainably support a **maximum population** of size k , with only **birth rate** as f .

Logistically, we know that:

1. As the population grows toward the capacity, the rate should decline to 0.
2. At the capacity, the rate should be 0.
3. Beyond k , the rate should be negative to decrease the population.

Assume $R(f)$ be the population change rate, then we have:

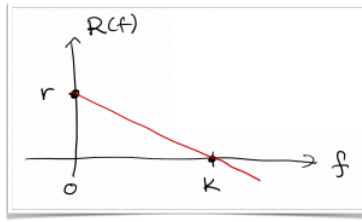


Figure 1: Rate of Change

Which can be expressed as:

$$R(f) = r - \frac{r}{k}f = r\left(1 - \frac{f}{k}\right) \quad (5)$$

So we have a new model, which is also called **logistic model of pop. growth**:

$$\frac{df}{dt} = Rf = r\left(1 - \frac{f}{k}\right)f \quad (6)$$

This model is simple enough to have an analytic solution:

$$f(t) = \frac{c}{c/k + (1 - c/k)\exp(-rt)} \quad (7)$$

3.3 Infection Spread

Now we come back to simulate the spread of infections. We have the following parameters:

1. k : Length of time of infection
2. τ : Probability of infection
3. $I(t)$: Infected, $I(t) \geq 0$
4. $S(t)$: Susceptible, $S(t) \leq 1$
5. $R(t)$: Recovered, $R(t) \leq 1$

Now we assume this is a **well-mixed** model, which means we ignore **spatial dimensions** and assume everyone is connected to everyone else.

Logistically, we know:

$$R(t) = 1 - I(t) - S(t) \quad (8)$$

$$\text{Recovery Rate} = \frac{I(t)}{k} \quad (9)$$

$$\text{Rate of New Infections} = \tau \cdot I(t) \cdot S(t) \quad (10)$$

Therefore, we have the equations:

$$\frac{dI(t)}{dt} = \tau I(t)S(t) - \frac{I(t)}{k} \quad (11)$$

$$\frac{dS(t)}{dt} = -\tau I(t)S(t) \quad (12)$$

$$\frac{dR(t)}{dt} = \frac{I(t)}{k} \quad (13)$$

If we want to reflect the idea that the infection lasts for k days, we need to introduce **delay** into the model:

$$\frac{dI(t)}{dt} = \tau I(t)S(t) - \tau I(t-k)S(t-k) \quad (14)$$

$$\frac{dS(t)}{dt} = -\tau I(t)S(t) \quad (15)$$

$$\frac{dR(t)}{dt} = \tau I(t-k)S(t-k) \quad (16)$$