Linearization Quals Problems

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1 Pendulum (Fall 2023)

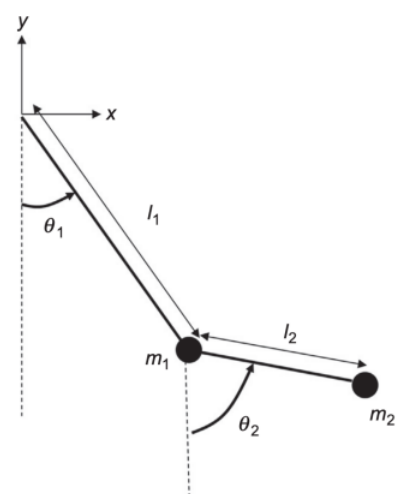

Figure 1: Pendulum

Consider a double pendulum, where m_i, l_i, and g are constants. The equations of motion for a double pendulum's θ_1 and θ_2 are

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2)\sin\theta_1 = 0, \tag{1}$$

$$l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g\sin\theta_2 = 0, \tag{2}$$

where $\theta_{1,2}$ are dependent variables in terms of time (overdots represent derivatives with respect to time).

(a) What are the variables $\mathbf{x}$ that make up the equivalent system of equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$? Hint: there are four of them.

(b) What are the four f_i's? In terms of the constants above and the variables x_i for $i = 1, \ldots, 4$ from (a). Show all work.

(c) Is your system of equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ linear or nonlinear? Explain.

(d) What are the fixed points of your $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$?

(e) This system also has four nullclines, one for each x_i. What are they?

(f) Computing the fixed points from these nullclines is not possible without a computer algebra system. Still, explain how you *would* do it, ignoring the actual nullclines you computed above.

1.1 Question 1

$$\boldsymbol{x} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{1}$$

1.2 Question 2

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{2}$$

Based on the observation, we can easily find out that:

$$f_1 = x_3, \quad f_2 = x_4 \tag{3}$$

Now, rearrange the previous given equations to get the expressions for $\dot{x_3}$ and $\dot{x_4}$ using $\boldsymbol{x}$:

$$f_3 = \dot{x}_3 = \frac{-m_2 l_2 \dot{x}_4 \cos(x_1 - x_2) - m_2 l_2 x_4^2 \sin(x_1 - x_2) - g(m_1 + m_2)\sin(x_1)}{(m_1 + m_2) l_1} \tag{4}$$

$$f_4 = \dot{x}_4 = \frac{-l_1 \dot{x}_3 \cos(x_1 - x_2) + l_1 x_3^2 \sin(x_1 - x_2) - g\sin(x_2)}{l_2} \tag{5}$$

1.3 Question 3

This system of equations is nonlinear, because there are *sin* and *cos* functions in the equations. Also, the terms with x^2 will make the equation nonlinear.

1.4 Question 4

Based on the definition of fixed point, we have:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} x_3 \\ x_4 \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Plug in these values, after simplification we have:

$$f_3 = \sin(x_1) = 0 \tag{7}$$

$$f_4 = \sin(x_2) = 0 \tag{8}$$

Therefore, the fix point $[\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]^T$ must satisfy the previous constraints. Therefore, θ_1, θ_2 could be 0 or multiples of π, respectively. The example fixed points include:

$$[0, 0, 0, 0]^T, \quad [\pi, \pi, 0, 0]^T, etc \tag{9}$$

1.5 Question 5

Based on the definition of the nullclines, they could be found by setting each $\dot{x_i} = 0$. Therefore, for $\dot{x_1} = 0$:

$$x_3 = 0 \tag{10}$$

For $\dot{x_2} = 0$:

$$x_4 = 0 \tag{11}$$

For $\dot{x_3} = 0$:

$$m_2 l_2 \ddot{x}_4 \cos(x_1 - x_2) + m_2 l_2 \dot{x}_4^2 \sin(x_1 - x_2) + g(m_1 + m_2)\sin(x_1) = 0 \tag{12}$$

For $\dot{x_4} = 0$:

$$l_1 \dot{x}_3 \cos(x_1 - x_2) - l_1 \dot{x}_3^2 \sin(x_1 - x_2) + g \sin(x_2) = 0 \tag{13}$$

1.6 Question 6

When solving the fixed point, just plug in 0 for $x_3, x_4, \dot{x_3}, \dot{x_4}$ into the system, solving the equations.

2 Integral Curve (Spring 2023)

Consider the equation

$$\ddot{x} + \alpha x + \beta x^3 = 0 \tag{14}$$

where $\alpha > 0$ and β can be positive or negative. Let $y \equiv \dot{x}$ for the purposes of constructing integral curves and phase portraits.

For $\beta > 0$:

(a) Find the singular point(s).

(b) Of what type is it (are they)? e.g., center, source, sink, spiral, saddle. If appropriate, is it (are they) stable or unstable?

(c) Find the equation(s) for the integral curve(s) (curves of constant "energy").

(d) Draw a qualitatively correct phase portrait.

For $\beta < 0$:

(e) Find the singular point(s).

(f) Of what type is it (are they)? e.g., center, source, sink, spiral, saddle. If appropriate, is it (are they) stable or unstable?

(g) Find the equation(s) for the integral curve(s) (curves of constant "energy").

(h) Draw a qualitatively correct phase portrait.

2.1 Question 1

Define:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{15}$$

Also:

$$\boldsymbol{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} \tag{16}$$

Now, based on the definition of fixed point, we have:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

Therefore:

$$\dot{x} = 0 \tag{18}$$

$$\ddot{x} = -\alpha x - \beta x^3 = -x(\alpha + \beta x^2) = 0 \tag{19}$$

For the normal dynamic system, we are typically interested in real-valued solutions because they correspond to physical states or positions of a system in most practical applications. Because we know that $\alpha > 0$ and $\beta > 0$ in this case, so the only real solution is $x = 0$, and the fixed point will be:

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

2.2 Question 2

To determine the type of the fixed point, we need to calculate the Jacobian matrix and eigenvalues at this fixed point. Recall that:

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \dot{x} \\ -\alpha x - \beta x^3 \end{bmatrix} \tag{21}$$

Therefore the Jacobian matrix will be:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & 0 \end{bmatrix} \tag{22}$$

Now take $[0, 0]^T$ as the fixed point, then:

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} \tag{23}$$

Now we calculate the eigenvalues of this Jacobian matrix:

$$\boldsymbol{J} - \lambda \boldsymbol{I} = \begin{bmatrix} -\lambda & 1 \\ -\alpha & -\lambda \end{bmatrix} \tag{24}$$

$$det(\boldsymbol{J} - \lambda \boldsymbol{I}) = \lambda^2 + \alpha = 0 \tag{25}$$

Therefore:

$$\lambda = \pm\sqrt{-\alpha} \tag{26}$$

Notice that:

$$\sqrt{-1} = \pm i \tag{27}$$

Therefore:

$$\lambda_1 = i\sqrt{\alpha}, \;\; \lambda_2 = -i\sqrt{\alpha} \tag{28}$$

Because both eigenvalues are pure imaginary values, so the fix point node is a center node, and it is neutrally stable.

2.3 Question 3

The total energy could be expressed as:

$$\text{Total Energy}(E) = \text{Kinetic Energy}(KE) + \text{Potential Energy}(V) \tag{29}$$

The specific kinetic energy could be expressed as:

$$\text{KE} = \frac{1}{2}v^2 \tag{30}$$

Here, x represents the distance, so the expression will be:

$$\text{KE} = \frac{1}{2}\dot{x}^2 = \frac{1}{2}y^2 \tag{31}$$

Now assume the potential energy as $V(x)$, then we have:

$$E(x,y) = \frac{1}{2}y^2 + V(x) \tag{32}$$

Based on the energy conservation, we have:

$$\frac{dE}{dt} = y\dot{y} + V'(x)\dot{x} = 0 \tag{33}$$

Plug in the expressions of $\dot{x}$, y and $\dot{y}$, we have:

$$y(-\alpha - \beta x^3) + V'(x)y = 0 \tag{34}$$

Therefore:

$$V'(x) = \alpha + \beta x^3 \tag{35}$$

Take the integral, we could have the potential energy function:

$$V(x) = \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4 + C \tag{36}$$

Therefore, the integral curves equation will be:

$$E(x,y) = \frac{1}{2}y^2 + \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4 + C \tag{37}$$

2.4 Question 4

Based on the answer from question 2, we already know the point is a center node. Now we want to know either the circle counterclockwise or clockwise. Recall the Jacobian matrix:

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} \tag{38}$$

For the circle, because in Jacobian matrix $bc < 0$, so the circle is clockwise.

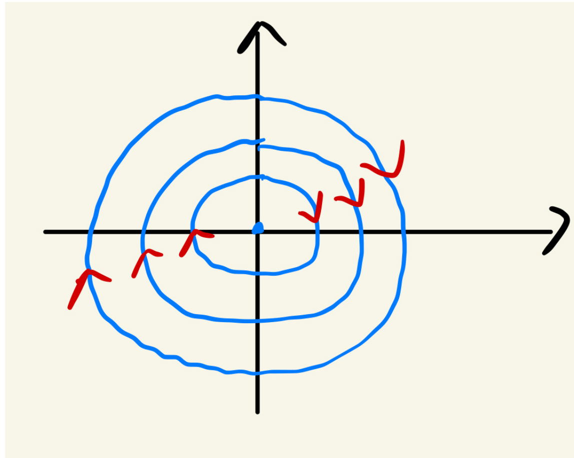

Figure 2: Clockwise Circle

2.5 Question 5

Similar with question 1, we have the expression:

$$\ddot{x} = -\alpha x - \beta x^3 = -x(\alpha + \beta x^2) = 0 \tag{39}$$

However, now because $\beta < 0$, so we have 3 solutions of x, and the fixed points will be:

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{-\frac{\alpha}{\beta}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{-\frac{\alpha}{\beta}} \\ 0 \end{bmatrix} \tag{40}$$

2.6 Question 6

The Jacobian matrix expression will be the same:

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -\alpha - 3\beta x^2 & 0 \end{bmatrix} \tag{41}$$

2.6.1 Fixed point as $[0, 0]^T$

First we choose the fixed point as $[0, 0]^T$, then:

$$\boldsymbol{J_1} = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} \tag{42}$$

So the eigenvalues will still be:

$$\lambda_1 = i\sqrt{\alpha}, \ \ \lambda_2 = -i\sqrt{\alpha} \tag{43}$$

This is a center node, neutrally stable.

2.6.2 Fixed point as $[\sqrt{-\frac{\alpha}{\beta}}, 0]^T$ and $[-\sqrt{-\frac{\alpha}{\beta}}, 0]^T$ (they are the same)

Now we choose the fixed points as $[\sqrt{-\frac{\alpha}{\beta}}, 0]^T$ and $-[\sqrt{-\frac{\alpha}{\beta}}, 0]^T$, then:

$$\boldsymbol{J_2} = \begin{bmatrix} 0 & 1 \\ 2\alpha & 0 \end{bmatrix} \tag{44}$$

$$\boldsymbol{J_2} - \lambda \boldsymbol{I} = \begin{bmatrix} -\lambda & 1 \\ 2\alpha & -\lambda \end{bmatrix} \tag{45}$$

$$det(\boldsymbol{J_2} - \lambda \boldsymbol{I}) = \lambda^2 - 2\alpha = 0 \tag{46}$$

Therefore, the eigenvalues will be:

$$\lambda_1 = \sqrt{2\alpha}, \ \ \lambda_2 = -\sqrt{2\alpha} \tag{47}$$

Therefore, both eigenvalues are real, with opposite sign, so the point is a saddle point, and it is unstable. To figure out which direction is stable, we need to calculate the eigenvector. For the unstable direction:

$$(\boldsymbol{J}_2 - \lambda_1 \boldsymbol{I})\boldsymbol{v_1} = \begin{bmatrix} -\sqrt{2\alpha} & 1 \\ 2\alpha & -\sqrt{2\alpha} \end{bmatrix} \boldsymbol{v_1} = 0 \tag{48}$$

Therefore:

$$\boldsymbol{v_1} = \begin{bmatrix} \sqrt{2\alpha} \\ 1 \end{bmatrix} \tag{49}$$

Similarly, we have the stable direction:

$$\boldsymbol{v_2} = \begin{bmatrix} -\sqrt{2\alpha} \\ 1 \end{bmatrix} \tag{50}$$

2.7 Question 7

The same as question 3.

2.8 Question 8

For the center node, it is easy, the same as question 4. Because in the Jacobian matrix, $bc < 0$, so the rotation direction is Clockwise For the saddle point nodes, we need to see how the points change close to the fixed points. Now because we know the stable direction at the fixed point, we can draw the phase diagram:

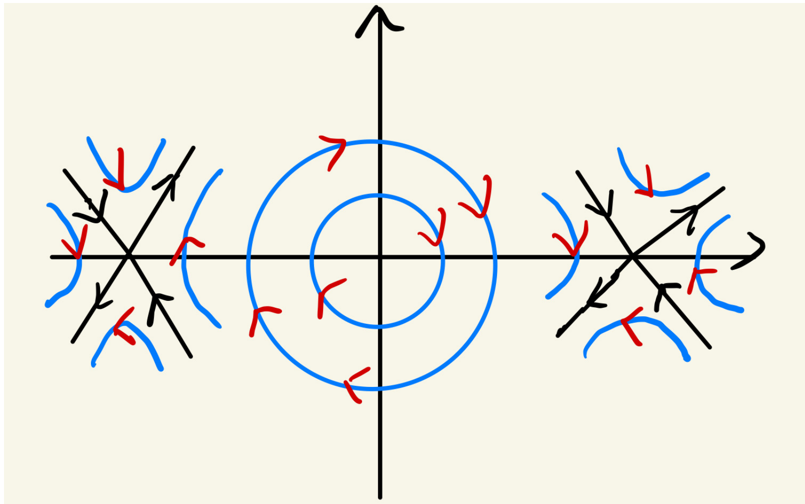

Figure 3: Final Phase Diagram

3 Nonuniform Oscillator (Fall 2022)

The dynamics of a *nonuniform* oscillator can be modeled as

$$\dot{\theta} = \omega - a\sin\theta, \tag{51}$$

where θ is an angle over a circle. If $a = 0$, we recover a uniform oscillator. It turns out that the flashes emitted by fireflies and even the human sleep–wake cycle can be modeled as nonuniform oscillators. Assume $\omega > 0$ and $a \geq 0$ for simplicity.

(a) Use linear stability to classify the fixed points of (1) for $a > \omega$.

(b) If $a < \omega$, determine the period of oscillation (call it $T(a)$) analytically, and graph what it looks like, roughly, for $0 \leq a \leq \omega$.

3.1 Question a

At the fixed point, we have:

$$\dot{\theta} = \omega - a\sin\theta = 0 \tag{52}$$

Because $a > \omega$, therefore (notice that arcsin conventionally refers only to the value within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$):

$$\theta_1^* = \arcsin\frac{\omega}{a}, \theta \in [0, \frac{\pi}{2}] \tag{53}$$

$$\theta_2^* = \pi - \arcsin\frac{\omega}{a}, \theta \in [\frac{\pi}{2}, \pi] \tag{54}$$

Because at fixed point, $\dot{\theta} = 0$, so we need to consider high order derivative to consider the stability:

$$\ddot{\theta} = -a\cos\theta \tag{55}$$

Therefore at θ_1^*, we have $-a\cos\theta_1^* < 0$, so this point is stable. At θ_2^*, we have $-a\cos\theta_2^* > 0$, so this point is unstable.

3.2 Question b

Recall the period formula:

$$T(a) = \int_0^{2\pi} \frac{d\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} \tag{56}$$

4 Bacteria Population (Spring 2022)

In this problem, you will consider a population of bacteria that produces a waste product that in high enough concentrations can be toxic to the bacterial population. The concentration of bacteria, represented by x, and the concentration of the waste product, represented by y, obey the following equations:

$$\frac{dx}{dt} = (a - by)x \tag{57}$$

$$\frac{dy}{dt} = cx - dy \tag{58}$$

Consider all four parameters (a, b, c, d) to be positive.

(a) Explain the model. What are the roles of the four parameters? What is the biological meaning of each of the terms on the right-hand sides of the equations?

(b) Find all equilibria (fixed points) of the system and analyze their stability.

(c) You should find in part (b) that a change in dynamics occurs for some condition. Explain the biological meaning of the condition.

4.1 Question a

The meaning of each parameter

1. a : This represents the natural growth rate of the bacterial population in the absence of any waste product.
2. b : This represents the inhibitory effect of the waste product on the bacterial population. As the concentration of the waste product y increases, the growth rate of the bacteria decreases.
3. c : This represents the rate at which the bacterial population produces the waste product.
4. d : This represents the natural degradation or removal rate of the waste product.

4.2 Question b

Define:

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \dot{\boldsymbol{x}} = \begin{bmatrix} (a-by)x \\ cx - dy \end{bmatrix} \tag{59}$$

To find the fixed points, we need:

$$f_1 = (a-by)x = 0 \tag{60}$$

$$f_2 = cx - dy = 0 \tag{61}$$

Therefore we have two fixed points:

$$\boldsymbol{x_1^*} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{x_2^*} = \begin{bmatrix} \frac{ad}{bc} \\ \frac{a}{b} \end{bmatrix} \tag{62}$$

Now get the general Jacobian matrix:

$$\boldsymbol{J} = \begin{bmatrix} a-by & -bx \\ c & -d \end{bmatrix} \tag{63}$$

1. When the fixed point is $[\frac{ad}{bc}, \frac{a}{b}]^T$:

$$\boldsymbol{J_1} = \begin{bmatrix} a & 0 \\ c & -d \end{bmatrix} \tag{64}$$

   Therefore, we have:

$$\lambda_1 = a, \ \lambda_2 = -d \tag{65}$$

   Because $a > 0, d > 0$, so this point is a saddle node.

2. When the fixed point is $[0, 0]^T$:

$$\boldsymbol{J_2} = \begin{bmatrix} 0 & -\frac{ad}{c} \\ c & -d \end{bmatrix} \tag{66}$$

   Therefore, we have:

$$\lambda = \frac{-d \pm \sqrt{d^2 - 4ad}}{2} \tag{67}$$

Therefore, there are several cases:

- If $d^2 - 4ad > 0$: two eigenvalues are negative, then this point is stable.

- If $d^2 - 4ad = 0$: now $d = 4a$, and we have:

$$\text{Tr}[\boldsymbol{J_2}] = -d \tag{68}$$

$$\det[\boldsymbol{J_2}] = ad \tag{69}$$

$$\Delta = (\text{Tr}[\boldsymbol{J_2}])^2 - 4\det[\boldsymbol{J_2}] = 0 \tag{70}$$

  Also, $\boldsymbol{J_2}$ has off-diagonal values, so this point is a stable degenerate point ($-d < 0$)

- If $d^2 - 4ad < 0$, then there are two complex eigenvalues, with same imaginary magnitude but different sign. Because the real part is negative, so it is a stable spiral. Because 'bc' in $\boldsymbol{J_2} = -ad < 0$, so the rotation direction is clockwise.

5 Inter-Species Interaction (Fall 2021)

Mutualism is an inter-species interaction that benefits all species involved.

(a) Explain why a model based on the classical Lotka-Volterra equations, such as

$$\frac{dN_1}{dt} = r_1 N_1 + a_1 N_1 N_2, \tag{71}$$
$$\frac{dN_2}{dt} = r_2 N_2 + a_2 N_1 N_2, \tag{72}$$

where the two species are N_1 and N_2 and r_1, r_2, a_1, and a_2 are all positive constants, would be a poor choice for a model of mutualism.

(b) Consider the model revision,

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} + b_{12}\frac{N_2}{K_1}\right) \tag{73}$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} + b_{21}\frac{N_1}{K_2}\right), \tag{74}$$

where r_1, r_2, K_1, K_2, b_{12}, b_{21} are all positive constants. Suppose we make the following substitutions: $u_1 = \frac{N_1}{K_1}$, $u_2 = \frac{N_2}{K_2}$, $\tau = r_1 t$, $\rho = \frac{r_2}{r_1}$, $a_{12} = b_{12}\frac{K_2}{K_1}$, and $a_{21} = b_{21}\frac{K_1}{K_2}$. Then we can rewrite the system with fewer parameters (note that we replace τ by t for simplicity):

$$\frac{du_1}{dt} = u_1(1 - u_1 + a_{12}u_2) = f_1(u_1, u_2) \tag{75}$$
$$\frac{du_2}{dt} = \rho u_2(1 - u_2 + a_{21}u_1) = f_2(u_1, u_2). \tag{76}$$

What biologically relevant fixed points (equilibrium values) exist for this system, and under what conditions?

(c) For the revised model in part (b), assess the stability of all biologically relevant fixed points you found. *Hint:* Use the trace-determinant method for the fixed points with the most complicated expressions (you may use it for the others as well if you like). There is no need to distinguish between nodes and foci/spirals.

(d) Based on your findings in part (c), what do you expect to happen to the population as $t \to \infty$ in most cases?

5.1 Question a

Notice that here all the parameters are positive constant, this does not make sense. The populations of the species should not increase constantly, they must be restricted by the environment and the overlimit of the population.

5.2 Question b

To find the fixed points, we need:

$$u_1(1 - u_1 + a_{12}u_2) = 0 \tag{77}$$

$$\rho u_2(1 - u_2 + a_{21}u_1) = 0 \tag{78}$$

From the first equaiton we know that $u_1 = 0$ or $u_2 = \frac{u_1 - 1}{a_{12}}$.

- When $u_1 = 0$, from the second equation:

$$\rho u_2(1 - u_2) = 0 \tag{79}$$

  So $u_2 = 0$ or $u_2 = 1$. So the fixed points are $[0, 0]^T$ and $[0, 1]^T$

- When $u_2 = \frac{u_1 - 1}{a_{12}}$, from the second equation:

$$\rho(\frac{u_1 - 1}{a_{12}})(1 - \frac{u_1 - 1}{a_{12}} + a_{21}u_1) = 0 \tag{80}$$

  Therefore, $u_1 = 1$ or $u_1 = \frac{-a_{12}-1}{a_{21}a_{12}-1}$, then $u_2 = 0$ or $u_2 = \frac{-a_{21}-1}{a_{21}a_{12}-1}$. And the fixed points will be $[1, 0]^T$ and $[\frac{-a_{12}-1}{a_{21}a_{12}-1}, \frac{-a_{21}-1}{a_{21}a_{12}-1}]^T$

5.3 Question c

The general Jacobian matrix is:

$$\boldsymbol{J} = \begin{bmatrix} 1 - 2u_1 + a_{12}u_2 & a_{12}u_1 \\ \rho a_{21}u_2 & \rho(1 - 2u_2 + a_{21}u_1) \end{bmatrix} \tag{81}$$

1. For the fixed point $[0, 0]^T$:

$$\boldsymbol{J_1} = \begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix} \tag{82}$$

Therefore the eigenvalues are:

$$\lambda_1 = 1, \;\; \lambda_2 = \rho \tag{83}$$

Two eigenvalues are real and positive, so it is a source node.

2. For the fixed point $[0, 1]^T$:

$$\boldsymbol{J_2} = \begin{bmatrix} 1 + a_{12} & 0 \\ \rho a_{21} & -\rho \end{bmatrix} \tag{84}$$

Therefore the eigenvalues are:

$$\lambda_1 = a_{12} + 1, \;\; \lambda_2 = -\rho \tag{85}$$

So two eigenvalues are real, one is positive and one is negative, so it is a Saddle node, unstable.

3. For the fixed point $[1, 0]^T$:

$$\boldsymbol{J_3} = \begin{bmatrix} -1 & a_{12} \\ 0 & \rho(1 + a_{21}) \end{bmatrix} \tag{86}$$

Therefore the eigenvalues are:

$$\lambda_1 = -1, \;\; \lambda_2 = \rho(1 + a_{21}) \tag{87}$$

So two eigenvalues are real, one is positive and one is negative, so it is a Saddle node, unstable.

4. For the fixed point $[\frac{-a_{12}-1}{a_{21}a_{12}-1}, \frac{-a_{21}-1}{a_{21}a_{12}-1}]^T$, the calculation is very complicated, but we can use the trace-determinant method (for the Jacobian matrix):

$$\lambda^2 - T\lambda + D = 0 \tag{88}$$

- **Stable Node:** $D > 0, \;\; T < 0$
- **Unstable Node:** $D > 0, \;\; T > 0$
- **Saddle Point:** $D < 0$
- **Stable Spiral:** $D > 0, \;\; T^2 - 4D < 0, \;\; T < 0$
- **Unstable Spiral:** $D > 0, \;\; T^2 - 4D < 0, \;\; T > 0$

- **Center:** $D > 0, \;\; T = 0$

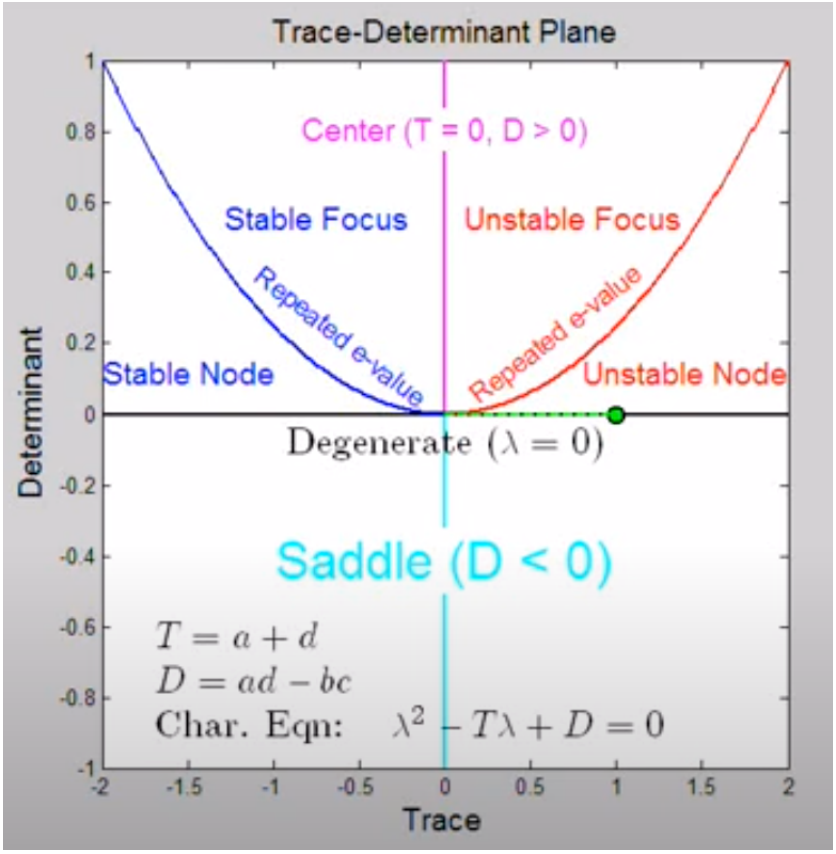

Figure 4: Trace Determinant Method

6 Non-Linear System (Fall 2018)

Consider the dynamical system with two state variables x and y whose behavior over time t is specified by the following equations:

$$x_t = x_{t-1}^2 - y_{t-1}$$

$$y_t = y_{t-1}x_{t-1} - y_{t-1}$$

Is this a linear or non-linear system? What are the equilibrium point(s) for this system? Complete a linear stability analysis of this system by determining the eigenvalues of the Jacobian matrix at each equilibrium point. Based on these calculations, discuss the stability of this system at each equilibrium point. Show all work in deriving your answers.

6.1 Solution

Because there exists a term with xy, so this system is nonlinear. At the equilibrium point, the values at t will be the same as $t-1$, therefore:

$$x^* = {x^*}^2 - y^* \tag{89}$$

$$y^* = y^* x^* - y^* \tag{90}$$

From the second equation, we know $y^* = 0$ or $x^* = 2$.

- When $y^* = 0$, then:

$$x^* = {x^*}^2 \tag{91}$$

  So $x^* = 0$ or $x^* = 1$.

- When $x^* = 2$, then from the first equation, $y^* = 2$.

Therefore, we have three fixed points $[0,0]^T$, $[1,0]^T$ and $[2,2]^T$.
Now get the general expression of Jacobian matrix:

$$\boldsymbol{J} = \begin{bmatrix} 2x^* - 1 & -1 \\ y^* & x^* - 2 \end{bmatrix} \tag{92}$$

- When the fixed point is $[0,0]^T$, then:

$$\boldsymbol{J_1} = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \tag{93}$$

  So the eigenvalues are:

$$\lambda_1 = -1, \ \ \lambda_2 = -2 \tag{94}$$

  Therefore this point is stable.

- When the fixed point is $[1,0]^T$, then:

$$\boldsymbol{J_2} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \tag{95}$$

  So the eigenvalues are:

$$\lambda_1 = 1, \ \ \lambda_2 = -1 \tag{96}$$

  So this point is a saddle point.

- When the fixed point is $[2,2]^T$, then:

$$\boldsymbol{J_2} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \tag{97}$$

  So the eigenvalues are:

$$\lambda_1 = 1, \ \ \lambda_2 = 2 \tag{98}$$

  So this point is a unstable node.